CORE Report No. 2004-03

DYNAMICS OF FLUID-CONVEYING BEAMS

by

J. N. REDDY and C.M. WANG

Centre for Offshore Research and Engineering

National University of Singapore

DYNAMICS OF FLUID-CONVEYING BEAMS:
Governing Equations and Finite Element Models

J. N. REDDY
Department of Mechanical Engineering
Texas A&M University
College Station, TX 77843-3123

C. M. WANG
Centre for Offshore Research and Engineering
Department of Civil Engineering
National University of Singapore
10 Kent Ridge Crescent, Singapore 119260

August 2004

DYNAMICS OF FLUID-CONVEYING BEAMS:
Governing Equations and Finite Element Models

J. N. REDDY[1]
Computational Mechanics Laboratory, Department of Mechanical Engineering
Texas A&M University, College Station, TX 77843-3123, U.S.A.

C. M. WANG
Centre for Offshore Research and Engineering
Department of Civil Engineering, National University of Singapore
10 Kent Ridge Crescent, Singapore 119260

Abstract – Equations of motion governing the deformation of fluid-conveying beams are derived using the kinematic assumptions of the (*a*) Euler–Bernoulli and (*b*) Timoshenko beam theories. The formulation accounts for geometric nonlinearity in the von Kármán sense and contributions of fluid velocity to the kinetic energy as well as to the body forces. Finite element models of the resulting nonlinear equations of motion are also presented.

Keywords – Analytical solution, Beams, Euler–Bernoulli beam theory, finite element method, fluid-conveying beams, Timoshenko beam theory, transverse motion, shear deformation.

1. INTRODUCTION

Fluid-conveying beams are found in many practical applications. They are encountered, for example, in the form of exhaust pipes in engines, stacks of flue gases, air-conditioning ducts, pipes carrying fluids (chemicals) in chemical and power plants, risers in offshore platforms, and tubes in heat exchangers and power plants. The fluid inside the pipe dynamically interacts with the pipe motion, possibly causing the pipe to vibrate.

Studies of fluid-conveying pipes have been reported in a number of papers. A survey of the subject by Paidoussis [1] indicates that more than 200 papers have been written in the open literature. Here we shall not attempt to review the vast literature on the dynamics of fluid-conveying pipes, but only cite few early papers and some recent papers that have direct bearing on the present paper.

[1] Author to whom correspondence should be sent. e-mail: *jnreddy@shakti.tamu.edu*; Tel: 001-979-862-2417; Fax: 001-979-862-3989.

The early contributions to the literature are due to Ashley and Haviland [2], Feodosyev [3], Housner [4], Benjamin [5], and Naguleswaran and Williams [6]. They all studied flexural vibration of a pipe conveying a fluid. Crandall et al. [7] and Dimarogonas and Haddad [8] developed the equations of motion of fluid conveying pipes using the kinematics of the Euler–Bernoulli beam theory.

Paidoussis and his coworkers [9-14] studied dynamics of pipes conveying fluid using both the Euler–Bernoulli beam theory and the Timoshenko beam theory (also see [15-20]). Semler, et al. [14] derived a complete set of geometrically nonlinear equations of motion of fluid conveying pipes. They accounted for large strains and assumed the kinematics of the Euler–Bernoulli beam theory. The equations derived in [14] are more complete than seen in most other papers and account for large strains and rotations (only in kinematic sense and not material sense). However, they are valid only for the Euler–Bernoulli beams without rotary inertia and are unduly complicated (and perhaps inconsistent because the stress-strain relations used do not account for the material density changes) for the analysis of fluid-conveying pipes, especially when the pipe does not undergo large deformation.

The present paper presents simple but complete derivation of the equations of motion of fluid-conveying pipes with small strains but moderate rotations. Derivations are presented for the Euler–Bernoulli beam theory and the Timoshenko beam theory and they are based on energy considerations (i.e., using the dynamic version of the principle of virtual displacements). The geometric nonlinearity in the von Kármán sense is included and contributions of fluid velocity to the kinetic energy as well as to the body forces are accounted for. The resulting nonlinear equations agree with those of Semler, et al. [14] for the small strains case, and the present equations include rotary inertia terms as well as transverse shear strains. Finite element models of the equations of motion are also developed. Numerical solutions using the finite element method will be presented in Part 2 of this paper to bring out the effect of transverse shear deformation in the pipe and fluid velocity on the transverse motion.

2. THE EULER–BERNOULLI BEAM THEORY

2.1 Displacements and Strains

The Euler–Bernoulli hypothesis requires that plane sections perpendicular to the axis of the beam before deformation remain (*a*) plane, (*b*) rigid (not deform), and (*c*) rotate such that they remain perpendicular to the (deformed) axis after deformation (see Reddy [21–23]). The assumptions amount to neglecting the Poisson effect and transverse strains. The bending of beams with moderately large rotations but with small strains can be derived using the displacement field

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_0}{\partial x}, \quad w(x,z,t) = w_0(x,t) \tag{2.1}$$

where (u, w) are the total displacements along the coordinate directions (x, z), and

u_0 and w_0 denote the axial and transverse displacements of a point on the neutral axis at time t.

Using the nonlinear strain-displacement relations and by omitting the large strain terms but retaining only the square of $\partial w_0/\partial x$ (which represents the rotation of a transverse normal line in the beam), we obtain

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 + z\left(-\frac{\partial^2 w_0}{\partial x^2}\right) \equiv \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \tag{2.2a}$$

where

$$\varepsilon_{xx}^0 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \ , \quad \varepsilon_{xx}^1 = -\frac{\partial^2 w_0}{\partial x^2} \tag{2.2b}$$

and all other strains are zero.

2.2 Virtual Work

We assume that the beam is hollow and subjected to a transverse distributed load of $q(x,t)$ along the length of the beam, and suppose that a fluid with velocity $\mathbf{v}(t)$ is being conveyed by the beam. The distributed load may include the weight of the beam as well as the fluid, and also the hydrostatic pressure if the beam is submerged in water. The forces on the beam due to the centripetal and tangential accelerations of the fluid can be accounted for in the potential energy of the loads (see Fig. 2.1). The fluid velocity $\mathbf{v}(t)$ also contributes to the kinetic energy of the system. Since we are primarily interested in deriving the equations of motion and the nature of the boundary conditions of the beam that experiences a displacement field of the form in Eq. (2.1), we will not consider specific geometric or force boundary conditions here.

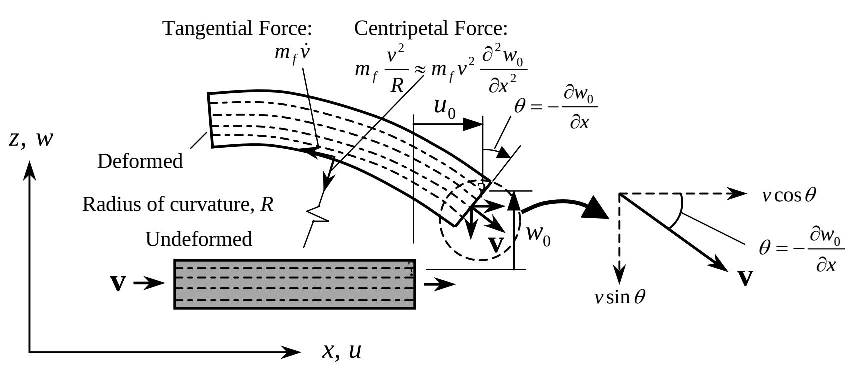

Fig. 2.1 An element of fluid-conveying beam in the Euler–Bernoulli beam theory.

The dynamic version of the principle of virtual displacements (i.e. Hamilton's principle for deformable bodies) is given by

$$\int_0^T \left[(\delta U - \delta V) - \delta K\right] dt = 0 \tag{2.3}$$

where δU is the virtual work due to internal forces, δV the virtual work due to external forces (including those of flowing fluid), and δK is the virtual kinetic energy in the beam as well as the fluid inside the beam. We have

$$\delta U = \int_0^L \int_{A_p} \sigma_{xx} \left(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)}\right) dAdx \tag{2.4a}$$

$$\begin{aligned}\delta V = \int_0^L \Big[& q\delta w_0 - m_f v^2 \frac{\partial^2 w_0}{\partial x^2}\left(\sin\theta\,\delta u_0 + \cos\theta\,\delta w_0\right) \\ & - m_f \dot{v}\left(\cos\theta\,\delta u_0 - \sin\theta\,\delta w_0\right)\Big] dx\end{aligned} \tag{2.4b}$$

$$\begin{aligned}\delta K = & \int_0^L \int_{A_p} \rho_p \left[\left(\dot{u}_0 - z\frac{\partial \dot{w}_0}{\partial x}\right)\left(\delta\dot{u}_0 - z\frac{\partial \delta\dot{w}_0}{\partial x}\right) + \dot{w}_0\delta\dot{w}_0\right] dAdx \\ & + \int_0^L \int_{A_f} \rho_f \left[\mathbf{v}\cdot\delta\mathbf{v} + z^2\left(\frac{\partial \dot{w}_0}{\partial x}\right)\left(\frac{\partial \delta\dot{w}_0}{\partial x}\right)\right] dAdx\end{aligned} \tag{2.4c}$$

where ρ_p is the mass density of the beam material; ρ_f is the mass density of the fluid inside the beam; A_p and A_f are the cross-sectional areas of the beam material and beam fluid passage, respectively; $m_f = \rho_f A_f$ and $m_p = \rho_p A_p$ are the mass densities of the fluid and beam, respectively, per unit lenght; and $\mathbf{v}$ is the fluid velocity vector (see Fig. 2.1)

$$\mathbf{v} = \left(v\cos\theta + \dot{u}_0\right)\hat{\mathbf{i}} + \left(-v\sin\theta + \dot{w}_0\right)\hat{\mathbf{j}}, \quad \theta = -\frac{\partial w_0}{\partial x} \tag{2.5}$$

2.3 Euler–Lagrange Equations

Substituting for $\delta\Pi$ and δK from Eqs. (2.4a) and (2.4b) into (2.3), we obtain

$$\begin{aligned}0 = & \int_0^T \int_0^L \Big\{ \left(N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)}\right) - m_p\left(\dot{u}_0\delta\dot{u}_0 + \dot{w}_0\delta\dot{w}_0\right) - (\hat{I}_p + \hat{I}_f)\frac{\partial \dot{w}_0}{\partial x}\frac{\partial \delta\dot{w}_0}{\partial x} \\ & - m_f\left[\left(v\cos\theta + \dot{u}_0\right)\ \delta\left(v\cos\theta + \dot{u}_0\right) + \left(v\sin\theta - \dot{w}_0\right)\ \delta\left(v\sin\theta - \dot{w}_0\right)\right]\Big\} dxdt \\ & - \int_0^L \left[q\delta w_0 - m_f v^2\frac{\partial^2 w_0}{\partial x^2}\left(\sin\theta\,\delta u_0 + \cos\theta\,\delta w_0\right) - m_f\dot{v}\left(\cos\theta\,\delta u_0 - \sin\theta\,\delta w_0\right)\right] dx\end{aligned} \tag{2.6a}$$

$$
\begin{aligned}
&= \int_0^T \int_0^L \Big[-\frac{\partial N_{xx}}{\partial x} + \left(m_p + m_f\right) \frac{\partial^2 u_0}{\partial t^2} + m_f v \sin\theta \left(\frac{\partial^2 w_0}{\partial x \partial t} + v \frac{\partial^2 w_0}{\partial x^2} \right) \\
&\qquad\qquad + m_f \dot{v} \cos\theta \Big] \delta u_0 \; dxdt \\
&+ \int_0^T \int_0^L \Big\{ -\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial}{\partial x} \left(\frac{\partial w_0}{\partial x} N_{xx} \right) + \left(m_p + m_f\right) \frac{\partial^2 w_0}{\partial t^2} - (\hat{I}_p + \hat{I}_f) \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \\
&\qquad\qquad - q + m_f v^2 \cos\theta \frac{\partial^2 w_0}{\partial x^2} - m_f \dot{v} \sin\theta \\
&\qquad + m_f v \Big[\cos\theta \left(2 \frac{\partial^2 w_0}{\partial x \partial t} - \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial x^2} \right) + \sin\theta \left(\frac{\partial^2 u_0}{\partial x \partial t} + \frac{\partial w_0}{\partial t} \frac{\partial w_0}{\partial x^2} \right) \Big] \Big\} \delta w_0 \; dxdt \\
&+ \int_0^T \Big\{ N_{xx} \delta u_0 - M_{xx} \frac{\partial \delta w_0}{\partial x} + \Big[\frac{\partial M_{xx}}{\partial x} + \frac{\partial w_0}{\partial x} N_{xx} - (\hat{I}_p + \hat{I}_f) \frac{\partial^3 w_0}{\partial x \partial t^2} \\
&\qquad\qquad - m_f v \left(\sin\theta \dot{u}_0 + \cos\theta \dot{w}_0 \right) \Big] \delta w_0 \Big\}_0^L dt
\end{aligned}
\tag{2.6b}
$$

where all the terms involving $[\ \cdot\ ]_0^T$ vanish on account of the assumption that all variations and their derivatives are zero at $t = 0$ and $t = T$, and the variables introduced in arriving at the last expression are defined as follows:

$$
\begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix} = \int_{A_p} \begin{Bmatrix} 1 \\ z \end{Bmatrix} \sigma_{xx} \; dA \tag{2.7a}
$$

$$
m_p = \int_{A_p} \rho_p \; dA = \rho_p A_p, \quad \hat{I}_p = \int_{A_p} \rho_p \, z^2 \; dA = \rho_p I_p \tag{2.7b}
$$

$$
m_f = \int_{A_f} \rho_f \; dA = \rho_f A_f, \quad \hat{I}_f = \int_{A_f} \rho_f \, z^2 \; dA = \rho_f I_f \tag{2.7c}
$$

Thus, the Euler–Lagrange equations of motion are

$$
-\frac{\partial N_{xx}}{\partial x} + \left(m_p + m_f\right) \frac{\partial^2 u_0}{\partial t^2} + m_f v \sin\theta \left(\frac{\partial^2 w_0}{\partial x \partial t} + v \frac{\partial^2 w_0}{\partial x^2} \right) + m_f \dot{v} \cos\theta = 0 \tag{2.8}
$$

$$
\begin{aligned}
&-\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial}{\partial x} \left(\frac{\partial w_0}{\partial x} N_{xx} \right) + \left(m_p + m_f\right) \frac{\partial^2 w_0}{\partial t^2} - (\hat{I}_p + \hat{I}_f) \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \\
&\quad + m_f v \Big[\cos\theta \left(2 \frac{\partial^2 w_0}{\partial x \partial t} - \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial x^2} \right) + \sin\theta \left(\frac{\partial w_0}{\partial t} \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x \partial t} \right) \Big] \\
&\qquad\qquad + m_f v^2 \cos\theta \frac{\partial^2 w_0}{\partial x^2} - m_f \dot{v} \sin\theta = q
\end{aligned}
\tag{2.9}
$$

Equations (2.8) and (2.9) represent coupled nonlinear equations among (u_0, w_0). To the authors' knowledge, this is the first time that the more complete version of equations of motion for fluid conveying beams are derived systematically for the case of small strains and moderate rotations.

2.4 Simplified Cases

The well-known equations governing Euler–Bernoulli beams without the fluid are obtained by setting $v = 0$ and $m_f = 0$ in Eqs. (2.8) and (2.9):

$$-\frac{\partial N_{xx}}{\partial x} + m_p \frac{\partial^2 u_0}{\partial t^2} = 0 \tag{2.10}$$

$$-\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x} N_{xx}\right) + m_p \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_p \frac{\partial^4 w_0}{\partial x^2 \partial t^2} = q \tag{2.11}$$

where the stress resultants N_{xx} and M_{xx} are related to the displacements (u_0, w_0) by

$$N_{xx} = E_p A_p \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right], \quad M_{xx} = -E_p I_p \frac{\partial^2 w_0}{\partial x^2} \tag{2.12}$$

If we assume that $\theta = -\partial w_0/\partial x$ is small compared to unity, then $\cos\theta \approx 1$ and $\sin\theta \approx \theta$, and Eqs. (2.8) and (2.9) become (for constant material and geometric parameters)

$$\left(m_p + m_f\right)\ddot{u}_0 + m_f \dot{v} - E_p A_p \left(u_0^{''} + w_0^{'} w_0^{''}\right) - m_f v w_0^{'} \left(\dot{w}_0^{'} + v w_0^{''}\right) = 0 \tag{2.13}$$

$$\begin{aligned}\left(m_p + m_f\right)\ddot{w}_0 - (\hat{I}_p + \hat{I}_f)\ddot{w}_0^{''} + m_f \dot{v} w_0^{'} + 2 m_f v \dot{w}_0^{'} + m_f v^2 w_0^{''} \\ + E_p I_p w_0^{''''} - E_p A_p \left[w_0^{''} u_0^{'} + 1.5 w_0^{''} (w_0^{'})^2 + w_0^{'} u_0^{''}\right] \\ - m_f v \left[\dot{u}_0 w_0^{''} + w_0^{'} \left(\dot{w}_0 w_0^{''} + \dot{u}_0^{'}\right)\right] = q\end{aligned} \tag{2.14}$$

where the prime ' denotes partial differentiation with respect to x and the superposed dot denotes partial differentiation with respect to time t. Most of the terms in the above equations agree with those derived by Semler, et al. [14]. The large strain terms and terms due to externally applied tension (T_0) and pressurization (P) are extra in [14]. On the other hand, Semler, et al. [14] do not account for the rotary inertia term and terms resulting from the approximation $\sin\theta \approx \theta$. It is not clear why the latter is neglected in [14].

By omitting all of the nonlinear terms, we obtain

$$\left(m_p + m_f\right)\ddot{u}_0 + m_f \dot{v} - E_p A_p u_0^{''} = 0 \tag{2.15}$$

$$\left(m_p + m_f\right)\ddot{w}_0 - (\hat{I}_p + \hat{I}_f)\ddot{w}_0^{''} + m_f \left(\dot{v} w_0^{'} + 2 v \dot{w}_0^{'} + v^2 w_0^{''}\right) + E_p I_p w_0^{''''} = q \tag{2.16}$$

The linearized equation of motion (2.16), without the rotary inertia term, can be found in the books by Crandall et al. [7] and Dimarogonas and Haddad [8]. They derived Eq. (2.16) using the Eulerian formulation for the fluid flow and neglecting the axial component of displacement [hence, Eq. (2.15) is omitted]. The first term in (2.16) denotes bulk acceleration, the second term denotes the bulk rotary inertia term, the third term is due to change in flow velocity, the fourth term is the Coriolis acceleration, and the fifth term is the contribution of centripetal acceleration. The linearized equations indicate that the axial motion is no longer damped while the transverse motion is damped by the velocity of the fluid.

3. THE TIMOSHENKO BEAM THEORY

3.1 Displacements and Strains

In the Timoshenko beam theory, the normality condition of the Euler–Bernoulli hypothesis is relaxed (i.e., the rotation is no longer equal to $-\partial w_0/\partial x$; see Fig. 3.1). The rotation of a transverse normal is treated as an independent variable. Consequently, the transverse shear strain γ_{xz} is no longer zero but a constant through the depth of the beam. The displacement field of the Timoshenko beam theory is (see Reddy [21])

$$u(x,z,t) = u_0(x,t) + z\phi(x,t), \quad w(x,z,t) = w_0(x,t) \tag{3.1}$$

where ϕ denotes the rotation of a transverse normal about the y axis.

The nonlinear strain-displacement relations are

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 + z\left(\frac{\partial \phi}{\partial x}\right) \equiv \varepsilon^0_{xx} + z\varepsilon^1_{xx} \tag{3.2a}$$

$$2\varepsilon_{xz} = \frac{\partial w_0}{\partial x} + \phi \equiv \gamma_{xz} \tag{3.2b}$$

where

$$\varepsilon^0_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \;, \quad \varepsilon^1_{xx} = \frac{\partial \phi}{\partial x} \tag{3.2c}$$

and all other strains are zero.

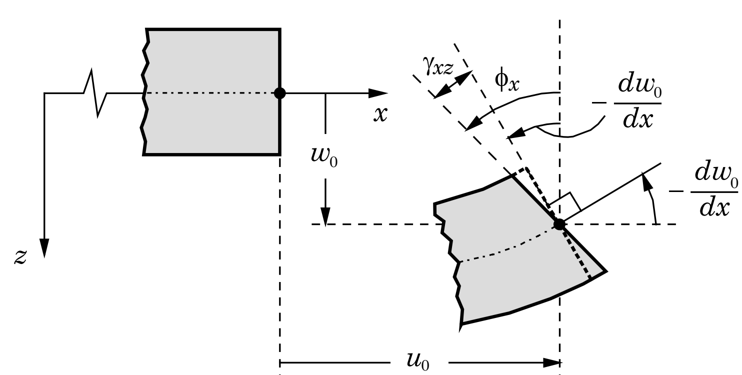

Fig. 3.1 Kinematics of the Timoshenko beam theory.

3.2 Virtual Work

The expressions for the total virtual work done and the virtual kinetic energy in the Timoshenko beam theory are given by

$$
\begin{aligned}
\delta\Pi = & \int_0^L \int_{A_p} \left[\sigma_{xx}\left(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)}\right) + \sigma_{xz}\gamma_{xz}\right] dAdx - \int_0^L q\delta w_0 \ dx \\
& + \int_0^L m_f v^2 \frac{\partial^2 w_0}{\partial x^2}\left(\sin\theta\,\delta u_0 + \cos\theta\,\delta w_0\right) dx \\
& + \int_0^L m_f \dot{v}\left(\cos\theta\,\delta u_0 - \sin\theta\,\delta w_0\right) dx
\end{aligned} \tag{3.3a}
$$

$$
\begin{aligned}
\delta K = & \int_0^L \int_{A_p} \rho_p \left[\left(\dot{u}_0 + z\dot{\phi}\right)\left(\delta\dot{u}_0 + z\delta\dot{\phi}\right) + \dot{w}_0\delta\dot{w}_0\right] dAdx \\
& + \int_0^L \int_{A_f} \rho_f \left(\mathbf{v}\cdot\delta\mathbf{v} + z^2\dot{\phi}\delta\dot{\phi}\right) dAdx
\end{aligned} \tag{3.3b}
$$

where $\mathbf{v}$ is the fluid velocity vector given in Eq. (2.5).

3.3 Euler–Lagrange Equations

Substituting for $\delta\Pi$ and δK from Eqs. (3.3a) and (3.3b) [with $\mathbf{v}$ given by Eq. (2.5)] into (2.3), we obtain

$$
\begin{aligned}
0 = \int_0^T \int_0^L \Big\{ & N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)} + Q_x\delta\gamma_{xz} - m_p\left(\dot{u}_0\delta\dot{u}_0 + \dot{w}_0\delta\dot{w}_0\right) - (\hat{I}_p + \hat{I}_f)\dot{\phi}\delta\dot{\phi} \\
& - q\delta w_0 + m_f v^2 \frac{\partial^2 w_0}{\partial x^2}\left(\sin\theta\,\delta u_0 + \cos\theta\,\delta w_0\right) + m\dot{v}\left(\cos\theta\,\delta u_0 - \sin\theta\,\delta w_0\right) \\
& - m_f\left[\left(v\cos\theta + \dot{u}_0\right)\ \delta\left(v\cos\theta + \dot{u}_0\right) + \left(v\sin\theta - \dot{w}_0\right)\ \delta\left(v\sin\theta - \dot{w}_0\right)\right]\Big\} dxdt
\end{aligned} \tag{3.4a}
$$

$$
\begin{aligned}
= & \int_0^T \int_0^L \Big[-\frac{\partial N_{xx}}{\partial x} + \left(m_p + m_f\right)\frac{\partial^2 u_0}{\partial t^2} + m_f v\sin\theta\left(\frac{\partial^2 w_0}{\partial x\partial t} + v\frac{\partial^2 w_0}{\partial x^2}\right) \\
& \quad + m_f\dot{v}\cos\theta\Big]\delta u_0 \ dxdt + \int_0^T \int_0^L \left[-\frac{\partial M_{xx}}{\partial x} + Q_x + (\hat{I}_p + \hat{I}_f)\frac{\partial^2\phi}{\partial t^2}\right]\delta\phi \ dxdt \\
& + \int_0^T \int_0^L \Big\{ -\frac{\partial Q_x}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x}N_{xx}\right) + \left(m_p + m_f\right)\frac{\partial^2 w_0}{\partial t^2} - q + m_f v^2\cos\theta\frac{\partial^2 w_0}{\partial x^2} \\
& \quad + m_f v\Big[\cos\theta\left(2\frac{\partial^2 w_0}{\partial x\partial t} - \frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial x^2}\right) + \sin\theta\left(\frac{\partial^2 u_0}{\partial x\partial t} + \frac{\partial w_0}{\partial t}\frac{\partial w_0}{\partial x^2}\right)\Big] \\
& \quad - m_f\dot{v}\sin\theta\Big\}\delta w_0 \ dxdt
\end{aligned}
$$

$$+\int_0^T \left[N_{xx}\delta u_0 + M_{xx}\delta\phi + Q_x\delta w_0 + \frac{\partial w_0}{\partial x}N_{xx} - (\hat{I}_p + \hat{I}_f)\frac{\partial^3 w_0}{\partial x \partial t^2} \right.$$
$$\left. - m_f v\left(\sin\theta\dot{u}_0 + \cos\theta\dot{w}_0\right)\delta w_0 \right]_0^L dt \tag{3.4b}$$

where

$$Q_x = K_s \int_{A_p} \sigma_{xz}\ dA \tag{3.4c}$$

K_s being the shear correction coefficient. For circular pipes with outer radius a and inner radius b, the shear correction coefficient is given by (see Cowper [24] and Wang, et al. [25])

$$K_s = \frac{6(1+\nu)(1+s)^2}{(7+6\nu)(1+s)^2 + (20+12\nu)s^2}, \quad s = \frac{b}{a} \tag{3.5}$$

The Euler–Lagrange equations of motion of the Timoshenko beam theory are

$$-\frac{\partial N_{xx}}{\partial x} + \left(m_p + m_f\right)\frac{\partial^2 u_0}{\partial t^2} + m_f v\sin\theta\left(\frac{\partial^2 w_0}{\partial x \partial t} + v\frac{\partial^2 w_0}{\partial x^2}\right) + m_f\dot{v}\cos\theta = 0 \tag{3.6}$$

$$-\frac{\partial Q_x}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x}N_{xx}\right) + \left(m_p + m_f\right)\frac{\partial^2 w_0}{\partial t^2}$$
$$+m_f v\left[\cos\theta\left(2\frac{\partial^2 w_0}{\partial x \partial t} - \frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial x^2}\right) + \sin\theta\left(\frac{\partial w_0}{\partial t}\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x \partial t}\right)\right]$$
$$+m_f v^2\cos\theta\frac{\partial^2 w_0}{\partial x^2} - m_f\dot{v}\sin\theta = q \tag{3.7}$$

$$-\frac{\partial M_{xx}}{\partial x} + Q_x + (\hat{I}_p + \hat{I}_f)\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{3.8}$$

Equations (3.6)–(3.8) represent coupled nonlinear equations among (u_0, w_0, ϕ), and they cannot be be found in the literature.

3.4 Simplified Cases

The equations governing Timoshenko beams without the fluid are obtained by setting $v = 0$ and $m_f = 0$ in Eqs. (3.6)–(3.8):

$$-\frac{\partial N_{xx}}{\partial x} + m_p\frac{\partial^2 u_0}{\partial t^2} = 0 \tag{3.9}$$

$$-\frac{\partial Q_x}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial w_0}{\partial x}N_{xx}\right) + m_p\frac{\partial^2 w_0}{\partial t^2} = q \tag{3.10}$$

$$-\frac{\partial M_{xx}}{\partial x} + Q_x + \hat{I}_p\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{3.11}$$

where the stress resultants (N_{xx}, M_{xx}, Q_x) are known in terms of the displacements (u_0, w_0, ϕ) by the relations

$$N_{xx} = E_p A_p \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right], \quad M_{xx} = E_p I_p \frac{\partial \phi}{\partial x}, \quad Q_x = G_p A_p K_s \left(\frac{\partial w_0}{\partial x} + \phi\right) \tag{3.12}$$

Under the assumption that $\theta = -\partial w_0/\partial x$ is small compared to unity ($\cos\theta \approx 1$ and $\sin\theta \approx \theta$), Eqs. (2.22)–(2.24) are reduced to (for constant material and geometric parameters)

$$\left(m_p + m_f\right)\ddot{u}_0 + m_f\dot{v} - E_p A_p\left(u_0^{''} + w_0^{'} w_0^{''}\right) - m_f v w_0^{'}\left(\dot{w}_0^{'} + v w_0^{''}\right) = 0 \tag{3.13}$$

$$\begin{aligned}\left(m_p + m_f\right)\ddot{w}_0 + m_f\dot{v}w_0^{'} + 2m_f v\dot{w}_0^{'} + m_f v^2 w_0^{''}\\ -G_p A_p K_s\left(\phi^{'} + w_0^{''}\right) - E_p A_p\left[w_0^{''}u_0^{'} + 1.5(w_0^{'})^2 w_0^{''} + w_0^{'}u_0^{''}\right]\\ -m_f v\left[\dot{u}_0 w_0^{''} + w_0^{'}\left(\dot{w}_0 w_0^{''} + \dot{u}_0^{'}\right)\right] = q\end{aligned} \tag{3.14}$$

$$(\hat{I}_p + \hat{I}_f)\ddot{\phi} - E_p I_p \phi^{''} + G_p A_p K_s\left(\phi + w_0^{'}\right) = 0 \tag{3.15}$$

The linear equations associated with (3.13)–(3.15) are

$$\left(m_p + m_f\right)\ddot{u}_0 + m_f\dot{v} - E_p A_p u_0^{''} = 0 \tag{3.16}$$

$$\left(m_p + m_f\right)\ddot{w}_0 + m_f\dot{v}w_0^{'} + 2m_f v\dot{w}_0^{'} + m_f v^2 w_0^{''} - G_p A_p K_s\left(\phi^{'} + w_0^{''}\right) = q \tag{3.17}$$

$$(\hat{I}_p + \hat{I}_f)\ddot{\phi} - E_p I_p \phi^{''} + G_p A_p K_s\left(\phi + w_0^{'}\right) = 0 \tag{3.18}$$

Most papers on fluid-conveying Timoshenko beams [17–19] do not give the governing equations but somehow account for various terms in (3.16)–(3.18).

4. FINITE ELEMENT MODELS

4.1 The Euler–Bernoulli Beam Model

The finite element model of the equations of motion (2.8) and (2.9) can be constructed using the virtual work statement (2.6*a*). The virtual work statement over a typical element (x_a, x_b) can be written as

$$\begin{aligned}0 = \int_0^T \int_{x_a}^{x_b} \Bigg\{ & E_p A_p \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]\left(\frac{\partial \delta u_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x}\right) + E_p I_p \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 \delta w_0}{\partial x^2}\\ & - \left(m_p + m_f\right)\left(\dot{u}_0\delta\dot{u}_0 + \dot{w}_0\delta\dot{w}_0\right) - (\hat{I}_p + \hat{I}_f)\frac{\partial \dot{w}_0}{\partial x}\frac{\partial \delta\dot{w}_0}{\partial x}\\ & - m_f v\left[\cos\theta\,\delta\dot{u}_0 - \left(\dot{u}_0\sin\theta + \dot{w}_0\cos\theta\right)\delta\theta - \sin\theta\delta\dot{w}_0\right] - q\delta w_0\\ & + m_f v^2\frac{\partial^2 w_0}{\partial x^2}\left(\sin\theta\,\delta u_0 + \cos\theta\,\delta w_0\right) + m_f\dot{v}\left(\cos\theta\,\delta u_0 - \sin\theta\,\delta w_0\right)\Bigg\} dxdt\end{aligned} \tag{4.1}$$

which is equivalent to the following two statements:

$$0=\int_0^T\int_{x_a}^{x_b}\Bigg\{E_pA_p\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]\frac{\partial \delta u_0}{\partial x}+\left(m_p+m_f\right)\ddot{u}_0\delta u_0$$
$$+m_f v\sin\theta\left(\frac{\partial^2 w_0}{\partial x\partial t}+v\frac{\partial^2 w_0}{\partial x^2}\right)\delta u_0+m_f\dot{v}\cos\theta\,\delta u_0\Bigg\}dxdt \tag{4.2}$$

$$0=\int_0^T\int_{x_a}^{x_b}\Bigg\{E_pA_p\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]\frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x}+E_pI_p\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 \delta w_0}{\partial x^2}\Bigg\}dxdt$$
$$+\int_0^T\int_{x_a}^{x_b}\Bigg\{\left(m_p+m_f\right)\ddot{w}_0\delta w_0+(\hat{I}_p+\hat{I}_f)\frac{\partial \ddot{w}_0}{\partial x}\frac{\partial \delta w_0}{\partial x}$$
$$+m_f v\left[-\left(\dot{u}_0\sin\theta+\dot{w}_0\cos\theta\right)\frac{\partial \delta w_0}{\partial x}+\cos\theta\,\frac{\partial^2 w_0}{\partial x\partial t}\delta w_0\right]$$
$$+m_f v^2\frac{\partial^2 w_0}{\partial x^2}\cos\theta\delta w_0-m_f\dot{v}\sin\theta\,\delta w_0-q\delta w_0\Bigg\}dxdt \tag{4.3}$$

We assume finite element approximations of the form (see Reddy [21–23])

$$u_0(x,t)=\sum_{j=1}^{2}u_j(t)\psi_j(x)\ ,\quad w_0(x,t)=\sum_{j=1}^{4}\bar{\Delta}_j(t)\varphi_j(x) \tag{4.4}$$

$$\bar{\Delta}_1(t)\equiv w_0(x_a,t),\quad \bar{\Delta}_2(t)\equiv-\theta(x_a,t),\quad \bar{\Delta}_3(t)\equiv w_0(x_b,t),\quad \bar{\Delta}_4(t)\equiv-\theta(x_b,t) \tag{4.5}$$

and $\psi_j(x)$ are the linear Lagrange interpolation functions, and $\varphi_j(x)$ are the Hermite cubic interpolation functions (see Fig. 4.1).

Substituting Eq. (4.5) for $u_0(x,t)$ and $w_0(x,t)$, and $\delta u_0(x)=\psi_i(x)$ and $\delta w_0(x)=\varphi_i(x)$ (to obtain the ith algebraic equation of the model) into the weak forms (4.3) and (4.4), we obtain

$$\begin{bmatrix}\mathbf{M}^{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{22}\end{bmatrix}\begin{Bmatrix}\ddot{\boldsymbol{\Delta}}^1\\ \ddot{\boldsymbol{\Delta}}^2\end{Bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{C}^{12}\\ \mathbf{C}^{21} & \mathbf{C}^{22}\end{bmatrix}\begin{Bmatrix}\dot{\boldsymbol{\Delta}}^1\\ \dot{\boldsymbol{\Delta}}^2\end{Bmatrix}+\begin{bmatrix}\mathbf{K}^{11} & \mathbf{K}^{12}\\ \mathbf{K}^{21} & \mathbf{K}^{22}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\Delta}^1\\ \boldsymbol{\Delta}^2\end{Bmatrix}=\begin{Bmatrix}\mathbf{F}^1\\ \mathbf{F}^2\end{Bmatrix} \tag{4.6a}$$

where

$$\Delta_i^1=u_i\ ,\quad \Delta_I^2=\bar{\Delta}_I \tag{4.6b}$$

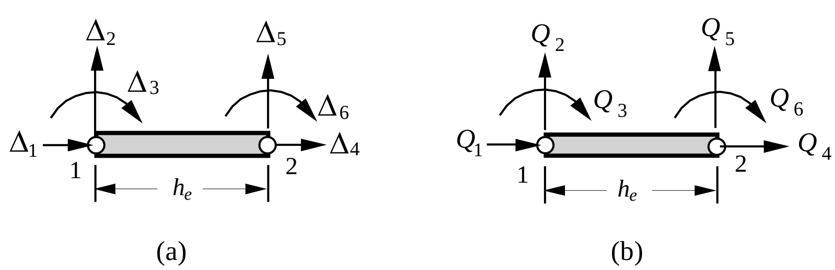

Fig. 4.1 A typical beam finite element with displacement and force degrees of freedom.

for $i = 1, 2$ and $I = 1, 2, 3, 4$. The element coefficients are

$$
\begin{aligned}
M_{ij}^{11} &= \int_{x_a}^{x_b} \Big(m_p + m_f\Big)\, \psi_i \psi_j \; dx \\
M_{IJ}^{22} &= \int_{x_a}^{x_b} \left[\Big(m_p + m_f\Big)\, \varphi_I \varphi_J + (\hat{I}_p + \hat{I}_f) \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} \right] dx \\
C_{iJ}^{12} &= \int_{x_a}^{x_b} m_f v \sin\theta \psi_i \frac{d\varphi_J}{dx} \; dx \approx - \int_{x_a}^{x_b} m_f v \frac{\partial w_0}{\partial x} \psi_i \frac{d\varphi_J}{dx} \; dx \\
C_{Ij}^{21} &= - \int_{x_a}^{x_b} m_f v \sin\theta \frac{d\varphi_I}{dx} \psi_j \; dx \approx \int_{x_a}^{x_b} m_f v \frac{\partial w_0}{\partial x} \frac{d\varphi_I}{dx} \psi_j \; dx \\
C_{Ij}^{22} &= \int_{x_a}^{x_b} m_f v \cos\theta \left(\frac{d\varphi_J}{dx} \varphi_I - \frac{d\varphi_I}{dx} \varphi_J \right) dx \approx \int_{x_a}^{x_b} m_f v \left(\frac{d\varphi_J}{dx} \varphi_I - \frac{d\varphi_I}{dx} \varphi_J \right) dx \\
K_{ij}^{11} &= \int_{x_a}^{x_b} E_p A_p \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} \; dx \\
K_{iJ}^{12} &= \frac{1}{2} \int_{x_a}^{x_b} \left[\left(E_p A_p \frac{\partial w_0}{\partial x} \right) \frac{d\psi_i}{dx} \frac{d\varphi_J}{dx} + m_f v^2 \sin\theta \psi_i \frac{d^2\varphi_J}{dx^2} \right] dx \\
&\approx \frac{1}{2} \int_{x_a}^{x_b} \left[\left(E_p A_p \frac{\partial w_0}{\partial x} \right) \frac{d\psi_i}{dx} \frac{d\varphi_J}{dx} - m_f v^2 \frac{\partial w_0}{\partial x} \psi_i \frac{d^2\varphi_J}{dx^2} \right] dx \\
K_{Ij}^{21} &= \int_{x_a}^{x_b} E_p A_p \frac{\partial w_0}{\partial x} \frac{d\varphi_I}{dx} \frac{d\psi_j}{dx} \; dx \;, \quad K_{Ij}^{21} = 2 K_{jI}^{12} \\
K_{IJ}^{22} &= \int_{x_a}^{x_b} E_p I_p \frac{d^2\varphi_I}{dx^2} \frac{d^2\varphi_J}{dx^2} \; dx + \frac{1}{2} \int_{x_a}^{x_b} \left[E_p A_p \left(\frac{\partial w_0}{\partial x} \right)^2 \right] \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} dx \\
&\quad + \int_{x_a}^{x_b} m_f v^2 \cos\theta \varphi_I \frac{d^2\varphi_J}{dx^2} \; dx \\
&\approx \int_{x_a}^{x_b} E_p I_p \frac{d^2\varphi_I}{dx^2} \frac{d^2\varphi_J}{dx^2} \; dx + \frac{1}{2} \int_{x_a}^{x_b} \left[E_p A_p \left(\frac{\partial w_0}{\partial x} \right)^2 \right] \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} dx \\
&\quad - \int_{x_a}^{x_b} m_f v^2 \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} \; dx + \int_{x_a}^{x_b} m_f \dot{v} \frac{d\varphi_J}{dx} \varphi_I \; dx \\
F_i^1 &= - \int_{x_a}^{x_b} m_f \dot{v} \cos\theta \psi_i \; dx + Q_i \approx - \int_{x_a}^{x_b} m_f \dot{v} \psi_i \; dx + Q_i \\
F_I^2 &= \int_{x_a}^{x_b} q \, \varphi_I \; dx - \int_{x_a}^{x_b} m_f \dot{v} \sin\theta \varphi_I \; dx + \hat{Q}_I \\
&\approx \int_{x_a}^{x_b} q \, \varphi_I \; dx + \int_{x_a}^{x_b} m_f \dot{v} \frac{\partial w_0}{\partial x} \varphi_I \; dx + \hat{Q}_I
\end{aligned}
\tag{4.7}
$$

for $(i, j = 1, 2)$ and $(I, J = 1, 2, 3, 4)$; Q_i and $\hat{Q}_I$ denote element end contributions from the extensional and bending terms, respectively. Note that the coefficient matrices $[K^{12}], [K^{21}]$ and $[K^{22}]$ are functions of the unknown $w_0(x, t)$. Stiffness coefficients are also given for the case in which $\cos\theta$ and $\sin\theta$ are approximated as $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.

4.2 The Timoshenko Beam Model

The finite element model of the Timoshenko beam equations can be constructed using the virtual work statement in Eq. (3.4a), where the axial force N_{xx}, the shear force Q_x, and bending moment M_{xx} are known in terms of the generalized displacements (u_0, w_0, ϕ) by Eq. (3.12).

The virtual work statement (3.4a) is equivalent to the following three statements:

$$
\begin{aligned}
0 = &\int_0^T \left[\int_{x_a}^{x_b} \left\{ E_p A_p \frac{\partial \delta u_0}{\partial x} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \right] \right\} dx - Q_1^e \delta u_0(x_a,t) - Q_4^e \delta u_0(x_b,t) \right] dt \\
&+ \int_0^T \int_{x_a}^{x_b} \left[(m_p + m_f)\ddot{u}_0 + m_f v \sin\theta \left(v\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial x \partial t} \right) + m_f \dot{v}\cos\theta \right] \delta u_0 \; dxdt
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
0 = &\int_0^T \left[\int_{x_a}^{x_b} \frac{\partial \delta w_0}{\partial x} \left\{ G_p A_p K \left(\frac{\partial w_0}{\partial x} + \phi \right) + E_p A_p \frac{\partial w_0}{\partial x} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \right] \right\} dx \right. \\
&\qquad \left. - \int_{x_a}^{x_b} \delta w_0 q \; dx - Q_2^e \delta w_0(x_a,t) - Q_5^e \delta w_0(x_b,t) \right] dt \\
&+ \int_0^T \int_{x_a}^{x_b} \left\{ \left[(m_p + m_f)\ddot{w}_0 + m_f v^2 \cos\theta \frac{\partial^2 w_0}{\partial x^2} + m_f v \cos\theta \frac{\partial^2 w_0}{\partial x \partial t} - m_f \dot{v}\sin\theta \right] \delta w_0 \right. \\
&\qquad \left. - m_f v \left(\sin\theta\, \dot{u}_0 + \cos\theta\, \dot{w}_0 \right) \frac{\partial \delta w_0}{\partial x} \right\} dxdt
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
0 = &\int_0^T \int_{x_a}^{x_b} \left[E_p I_p \frac{\partial \delta\phi}{\partial x}\frac{\partial \phi}{\partial x} + G_p A_p K \delta\phi \left(\frac{\partial w_0}{\partial x} + \phi \right) + (\hat{I}_p + \hat{I}_f)\ddot{\phi}\, \delta\phi \right] dxdt \\
&- \int_0^T \left[Q_3^e\, \delta\phi(x_a,t) + Q_6^e\, \delta\phi(x_b,t) \right] dt
\end{aligned} \tag{4.10}
$$

where δu_0, δw_0, and $\delta\phi$ are the virtual displacements. The Q_i^e have the same physical meaning as in the Euler–Bernoulli beam element, and their relationship to the horizontal displacement u_0, transverse deflection w_0, and rotation ϕ is

$$
\begin{aligned}
&Q_1^e(t) = -N_{xx}(x_a,t), \qquad Q_4^e = N_{xx}(x_b,t) \\
&Q_2^e(t) = -\left[Q_x + N_{xx}\frac{\partial w_0}{\partial x} \right]_{x=x_a}, \quad Q_5^e = \left[Q_x + N_{xx}\frac{\partial w_0}{\partial x} \right]_{x=x_b} \\
&Q_3^e(t) = -M_{xx}(x_a,t), \qquad Q_6^e(t) = M_{xx}(x_b,t)
\end{aligned} \tag{4.11}
$$

An examination of the virtual work statements (4.10a)–(4.10c) suggests that $u_0(x,t)$, $w_0(x,t)$, and $\phi(x,t)$ are the primary variables and therefore must be carried as nodal degrees of freedom. In general, u_0, w_0, and ϕ need not be approximated by polynomials of the same degree. However, the approximations should be such that possible deformation modes (i.e. kinematics) are represented correctly.

Suppose that the displacements are approximated as

$$u_0(x,t) = \sum_{j=1}^{m} u_j^e(t)\psi_j^{(1)}(x), \quad w_0(x,t) = \sum_{j=1}^{n} w_j^e(t)\psi_j^{(2)}(x), \quad \phi(x) = \sum_{j=1}^{p} s_j^e(t)\psi_j^{(3)}(x) \tag{4.12}$$

where $\psi_j^{(\alpha)}(x)$ ($\alpha = 1, 2, 3$) are Lagrange interpolation functions of degree $(m-1)$, $(n-1)$, and $(p-1)$, respectively. At the moment, the values of m, n, and p are arbitrary, that is, arbitrary degree of polynomial approximations of u_0, w_0, and ϕ may be used. Substitution of (4.12) for u_0, w_0, and ϕ, and $\delta u_0 = \psi_i^{(1)}$, $\delta w_0 = \psi_i^{(2)}$, and $\delta\phi_x = \psi_i^{(3)}$ into Eqs. (4.10a)–(4.10c) yields the finite element model

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{33} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{w}} \\ \ddot{\mathbf{s}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{C}^{12} & \mathbf{0} \\ \mathbf{C}^{21} & \mathbf{C}^{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{w}} \\ \dot{\mathbf{s}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{0} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ \mathbf{0} & \mathbf{K}^{32} & \mathbf{K}^{33} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{w} \\ \mathbf{s} \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \end{Bmatrix} \tag{4.13}$$

where

$$\begin{aligned}
M_{ij}^{11} &= \int_{x_a}^{x_b} (m_p + m_f)\psi_i^{(1)}\psi_j^{(1)}\, dx, \quad M_{ij}^{22} = \int_{x_a}^{x_b} (m_p + m_f)\psi_i^{(2)}\psi_j^{(2)}\, dx \\
M_{ij}^{33} &= \int_{x_a}^{x_b} (\hat{I}_p + \hat{I}_f)\psi_i^{(3)}\psi_j^{(3)}\, dx \\
C_{ij}^{12} &= \int_{x_a}^{x_b} m_f v \sin\theta \psi_i^{(1)} \frac{d\psi_j^{(2)}}{dx}\, dx \approx -\int_{x_a}^{x_b} m_f v \frac{\partial w_0}{\partial x}\psi_i^{(1)}\frac{d\psi_j^{(2)}}{dx}\, dx \\
C_{ij}^{21} &= -\int_{x_a}^{x_b} m_f v \sin\theta \frac{d\psi_i^{(2)}}{dx}\psi_j^{(1)}\, dx \approx \int_{x_a}^{x_b} m_f v \frac{\partial w_0}{\partial x}\frac{d\psi_i^{(2)}}{dx}\psi_j^{(1)}\, dx \\
C_{ij}^{22} &= \int_{x_a}^{x_b} m_f v \cos\theta \left(\psi_i^{(2)}\frac{d\psi_j^{(2)}}{dx} - \frac{d\psi_i^{(2)}}{dx}\psi_j^{(2)}\right) dx \\
&\approx \int_{x_a}^{x_b} m_f v \left(\psi_i^{(2)}\frac{d\psi_j^{(2)}}{dx} - \frac{d\psi_i^{(2)}}{dx}\psi_j^{(2)}\right) dx \\
K_{ij}^{11} &= \int_{x_a}^{x_b} E_p A_p \frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}\, dx, \quad K_{ij}^{21} = \int_{x_a}^{x_b} E_p A_p \frac{\partial w_0}{\partial x}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx}\, dx \\
K_{ij}^{12} &= \int_{x_a}^{x_b} \left[\frac{E_p A_p}{2}\frac{\partial w_0}{\partial x}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx} + m_f v^2 \sin\theta \psi_i^{(1)}\frac{d^2\psi_j^{(2)}}{dx^2}\right] dx
\end{aligned}$$

$$\approx \int_{x_a}^{x_b} \left[\frac{E_p A_p}{2} \frac{\partial w_0}{\partial x} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} - m_f v^2 \frac{\partial w_0}{\partial x} \psi_i^{(1)} \frac{d^2 \psi_j^{(2)}}{dx^2} \right] dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} G_p A_p K_s \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \, dx + \frac{1}{2} \int_{x_a}^{x_b} E_p A_p \left(\frac{dw_0}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \, dx$$

$$+ \int_{x_a}^{x_b} m_f v^2 \cos\theta \psi_i^{(2)} \frac{d^2 \psi_j^{(2)}}{dx^2} \, dx$$

$$\approx \int_{x_a}^{x_b} G_p A_p K_s \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \, dx + \frac{1}{2} \int_{x_a}^{x_b} E_p A_p \left(\frac{dw_0}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \, dx$$

$$- \int_{x_a}^{x_b} m_f v^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \, dx + \int_{x_a}^{x_b} m_f \dot{v} \psi_i^{(2)} \frac{d\psi_j^{(2)}}{dx} \, dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} G_p A_p K_s \frac{d\psi_i^{(2)}}{dx} \; \psi_j^{(3)} \, dx = K_{ji}^{32}$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(E_p I_p \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + G_p A_p K_s \psi_i^{(3)} \psi_j^{(3)} \right) \, dx$$

$$F_i^1 = - \int_{x_a}^{x_b} m_f \dot{v} \cos\theta \psi_i^{(1)} \, dx + Q_1^e \psi_i^{(1)}(x_a) + Q_4^e \psi_i^{(1)}(x_b)$$

$$\approx - \int_{x_a}^{x_b} m_f \dot{v} \psi_i^{(1)} \, dx + Q_1^e \psi_i^{(1)}(x_a) + Q_4^e \psi_i^{(1)}(x_b)$$

$$F_i^2 = \int_{x_a}^{x_b} \left(q + m_f \dot{v} \sin\theta \right) \psi_i^{(2)} \, dx + Q_2^e \psi_i^{(2)}(x_a) + Q_5^e \psi_i^{(2)}(x_b)$$

$$\approx \int_{x_a}^{x_b} \psi_i^{(2)} q \, dx + Q_2^e \psi_i^{(2)}(x_a) + Q_5^e \psi_i^{(2)}(x_b)$$

$$F_i^3 = Q_3^e \psi_i^{(3)}(x_a) + Q_6^e \psi_i^{(3)}(x_b) \tag{4.14}$$

The choice of the approximation functions $\psi_i^{(\alpha)}$ dictates different finite element models. When all field variables are interpolated with linear functions, the element is known to experience shear and membrane locking. To avoid shear and membrane locking (see Reddy [23]), one must use reduced integration to evaluate the transverse shear coefficients as well as the nonlinear terms.

5. PRELIMINARY NUMERICAL RESULTS

5.1 Static Nonlinear Analysis

First, we present results for the case of geometrically nonlinear analysis to illustrate the effect of transverse shear deformation on the nonlinear deflections. We use linear approximation of the axial displacement u_0 and Hermite cubic approximation of w_0 in the Euler–Bernoulli beam theory. In the Timoshenko beam theory, linear interpolation of both u_0 and w_0 is used. Reduced integration is used to alleviate the shear and membrane locking (see Reddy [22, 23]).

In the static nonlinear analysis, solid beams of rectangular section are used ($b = 1$, $L = 100$, and $L/h = 10$ or $L/h = 100$). The load parameter ($P = q_0/\Delta q$) versus nondimensional deflection ($\bar{w} = wEh^3/\Delta q L^3$) plots for a clamped-clamped (i.e., both ends are clamped) beam are shown in Fig. 5.1 (Δq is the load increment). The beam is loaded with uniformly distributed load of intensity q_0. A mesh of two elements (EBT or TBT elements) in the half beam are used. The effect of shear deformation is to increase the deflection (i.e., the kinematics of the Timoshenko beam theory make the beam more flexible). The load parameter ($P = F_0 L^2/EI$) versus nondimensional deflection (w/L) plots for a are shown in Fig. 5.2 for a cantilever beam (two Timoshenko beam elements in the full beam are used) under a point load F_0 at the free end.

5.2 Linear Transient Analysis

Linear transient analysis is presented using a rectangular channel cross-section beams with the following data (see Fig. 5.3)

$$B = 1.5 \times 10^{-3}\,\mathrm{m}, \quad H = 10.0 \times 10^{-3}\,\mathrm{m}, \quad b = 7 \times 10^{-3}\,\mathrm{m}, \quad h = 1.5 \times 10^{-3}\,\mathrm{m}$$

$$E = 2.5 \times 10^{8}\,\mathrm{GPa}, \quad G = 10^{8}\,\mathrm{GPa}, \quad \rho_b = 850\ \mathrm{kg/m}^3, \quad \rho_f = 10^3\ \mathrm{kg/m}^3$$

Figure 5.3 contains plots of the center deflection versus time for a simply supported beam under the weight of the beam and fluid but for $v = 0$, while Fig. 5.4 contains plots for $v = 1$ and $v = 2$. Clearly, the fluid has the effect of increasing the period of vibration (or reducing the frequency of oscillation). Additional investigation into the parametric effects of the material density and fluid density as well as the magnitude of the velocity is warranted. These results will appear elsewhere.

6. CONCLUDING REMARKS

In this paper the complete set of equations of motion governing fluid-conveying beams are derived using the dynamic version of the principle of virtual displacements. Equations for both the Euler–Bernoulli and Timoshenko beam theories are developed, and they account for the von Kármán nonlinear strains, rotary inertia, forces due to the flowing fluid in the beam, and kinetic energy of the flowing fluid. The resulting equations of motion contain all of the terms derived by others in the literature for the small strain case, but they also contain additional terms that were neglected. Finite element models of the governing equations of both theories are also presented. Preliminary numerical results are presented but more complete set of results will appear in a separate report.

Acknowledgement. The first author gratefully acknowledges the support of Department of Civil Engineering at the National University of Singapore for his stay as the Visiting Professor.

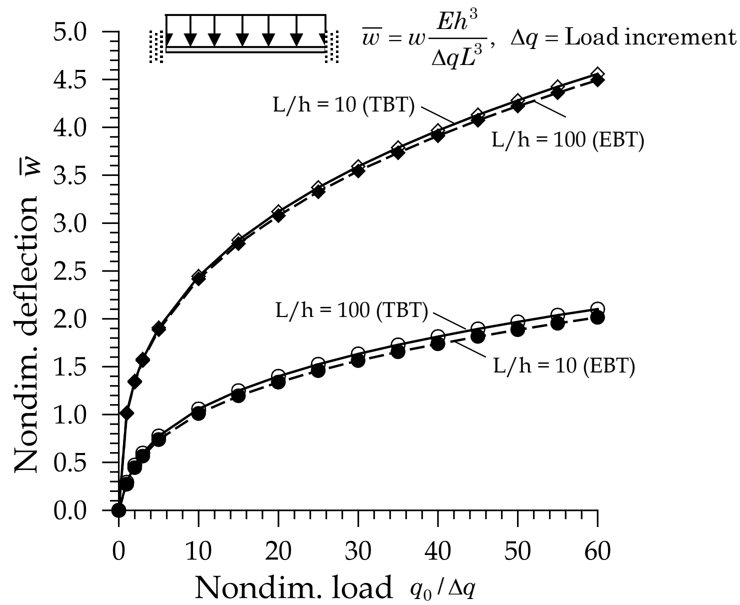

Fig. 5.1 Load versus deflection curves for a clamped beam under uniform transverse load.

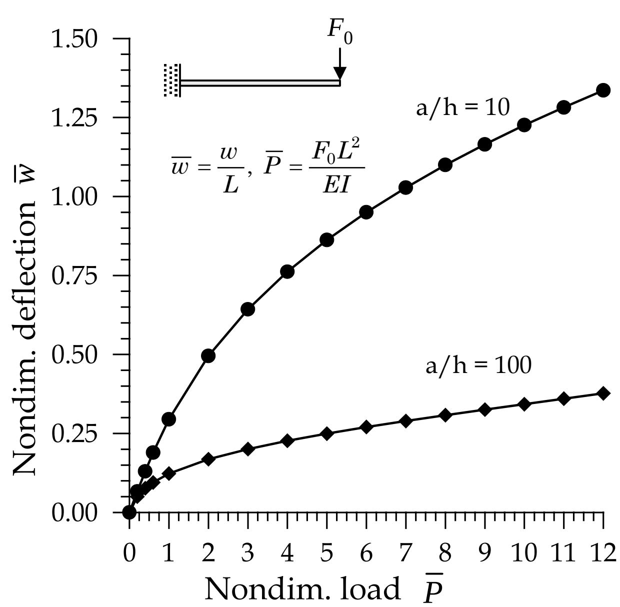

Fig. 5.2 Load versus deflection curves for a cantilever beam with an end point load.

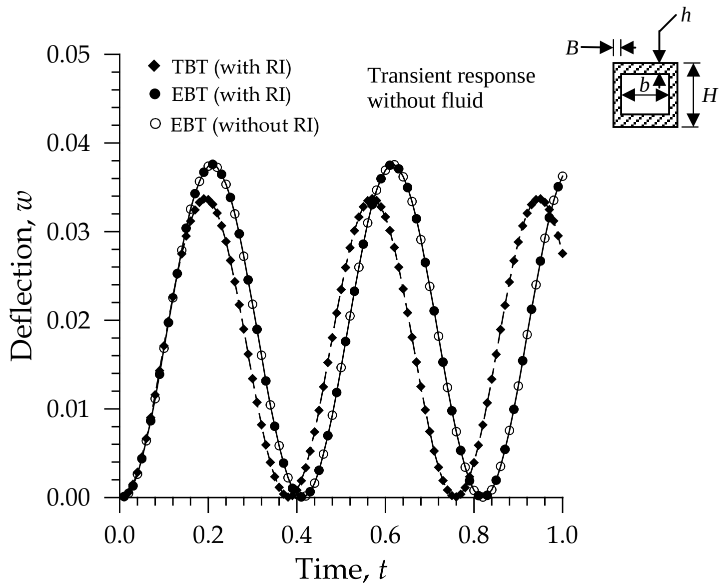

Fig. 5.3 Center deflection versus time for a simply supported beam under uniform transverse load ($v = 0$).

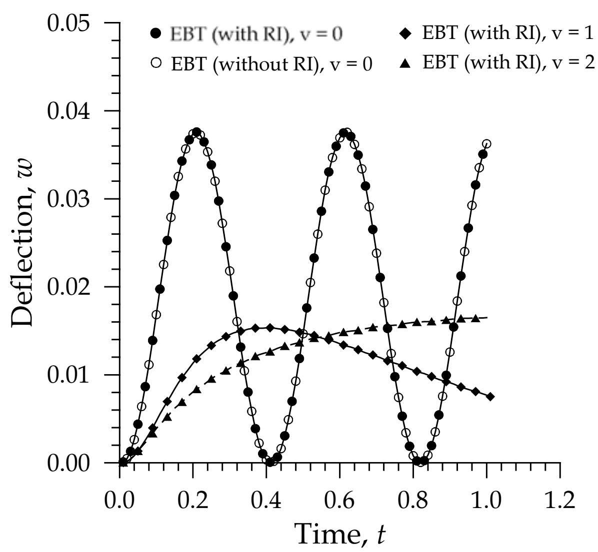

Fig. 5.4 Center deflection versus time for a simply supported beam under uniform transverse load ($v = 1$ m/s and $v = 2$ m/s).

REFERENCES

1. Paidoussis, M. P., "Pipes Conveying Fluid: A Model Dynamical Problem," *Proceedings of the Canadian Congress of Applied Mechanics,* Winnipeg, Manitoba, Canada, pp. 1–33, 1991; published as: Paidoussis, M. P. and Li, G. X., "Pipes Conveying Fluid: A Model Dynamical Problem," *Journal of Fluids and Structures,* **7**, 137–204, 1993.
2. Ashley, H. and Haviland, G., "Bending Vibrations of a Pipe Line Containing Flowing Fluid," *Journal of Applied Mechanics,* **17**, 229–232, 1950.
3. Feodosyev, V. I., "On the Vibrations and Stability of a Pipe Conveying a Fluid," (in Russian) *Engineer Book,* **10**, 169–170, 1951.
4. Housner, G. W., "Bending Vibrations of a Pipe Line Containing Flowing Fluid," *Journal of Applied Mechanics,* **19**, 205–208, 1952.
5. Benjamin, T. B., "Dynamics of a System Articulated Pipes Conveying Fluid: 1. Theory," *Proceedings of the Royal Society of London,* Series A, **261**, 457–486, 1961.
6. Naguleswaran, S. and Williams, C. J. H., "Lateral Vibration of a Pipe Conveying a Fluid," *Journal of Mechanical Engineering Science,* **10**, 228–238, 1968.
7. Crandal, S. H., Karnopp, D. C., Kurtz, Jr., E. F., and Pridmore–Brown, D. C., *Dynamics of Mechanical and Electromechanical Systems,* McGraw-Hill, New York, 1968 (see pp. 390–395).
8. Dimarogonas, A. D. and Haddad, S., *Vibration for Engineers*, Prentice-Hall, Englewood Cliffs, NJ, 1992 (see p. 446).
9. Paidoussis, M. P., "Dynamics of Flexible Slender Cylinders in Axial Flow," *Journal of Fluid Mechanics,* **26**, 717–736, 1966.
10. Paidoussis, M. P., "Dynamics of Tubular Cantilevers Conveying Fluid," *Journal of Mechanical Engineering Science,* **12**, 85–103, 1970.
11. Paidoussis, M. P. and Issid, T. D., "Dynamic Stability of Pipes Conveying Fluid," *Journal of Sound and Vibration,* **33**, 267–294, 1974.
12. Paidoussis, M. P. and Laithier, T. D., "Dynamics of Timoshenko Beam Conveying Fluid," *Journal of Mechanical Engineering Science,* **18**, 210–220, 1976.
13. Paidoussis, M. P., Luu, T. P., and Laithier, B. E., "Dynamics of Finite-Length Tubular Beams Conveying Fluid," *Journal of Sound and Vibration,* **106**, 311–331, 1986.
14. Semler, C., Li, G. X., and Paidoussis, M. P., "The Non-Linear Equations of Motion of Pipes Conveying Fluid," *Journal of Sound and Vibration,* **169**, 577–599, 1994.
15. Dupis, C. and Rousselet, J., "The Equations of Motion of Curved Pipes Conveying Fluid," *Journal of Sound and Vibration,* **153**(3), 473–489, 1994.

16. Pramila, A. and Laukkanen, J., "Dynamics and Stability of Short Fluid-Conveying Timoshenko Element Pipes," *Journal of Sound and Vibration,* **144**(3), 421–425, 1991.

17. Chu, C. L. and Lin, Y. H., "Finite Element Analysis of Fluid-Conveying Timoshenko Pipes," *Shock and Vibration,* **2**, 247–255, 1995.

18. Lin, Y.-H. and Tsai, Y.-K., "Nonlinear Vibrations of Timoshenko Pipes Conveying Fluid," *International Journal of Solids and Structures,* **34** (23), 2945–2956, 1997.

19. Zhang, Y. L., Gorman, D. G., and Reese, J. M. "Analysis of the Vibration of Pipes Conveying Fluid," *Proceedings of the Institution of Mechanical Engineers,* **213,** Part C, 849–859, 1999.

20. Lee, U. and Oh, H., "The Spectral Element Model for Pipelines Conveying Internal Steady Flow," *Engineering Structures,* **25** (23), 1045–1055, 2003.

21. Reddy, J. N., *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley, New York, 2002.

22. Reddy, J. N., *An Introduction to the Finite Element Method,* 2nd ed., McGraw-Hill, New York, 1993 (3rd ed. to appear in 2005).

23. Reddy, J. N., *An Introduction to Nonlinear Finite Element Analysis,* Oxford University Press, Oxford, UK, 2004.

24. Cowper, G. R., "The Shear Coefficient in Timoshenko's Beam Theory," *Journal of Applied Mechanics,* **33** 335–340, 1966.

25. Wang, C. M., Wang, C. Y., and Reddy, J. N., *Exact Solutions for Buckling of Structural Members,* CRC Press, Boca Raton, Florida, 2004.