A priori inverse operator estimation for guaranteed error estimate

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Purpose of this talk

Let $\Omega \subset \mathbf{R}^N \ \ (N = 1, 2)$ is convex polygonal domain.

We consider Drichlet boundary value problems of the form

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here, u : Exact solution, $\hat{u}$: Computational result,

'Guaranteed' Error Estimate:

$$\|u - \hat{u}\|_X \leqq Const.$$

X : suitable functional space,
$Const$: computable.

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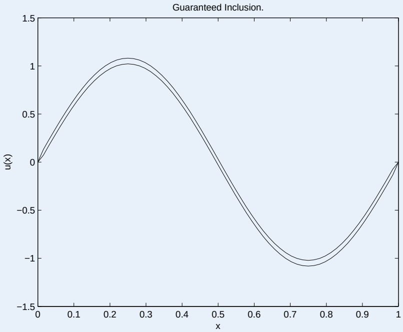
'Guaranteed' error proves...

- Existence of the exact solution.
- (Local) Uniqueness.
- Every error is included.
    - Discretization error.
    - Rounding error.

$\Rightarrow$ Mathematically guaranteed error by reliable computing.



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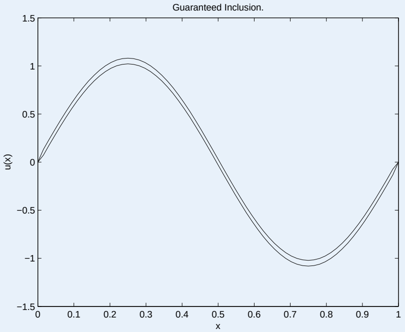

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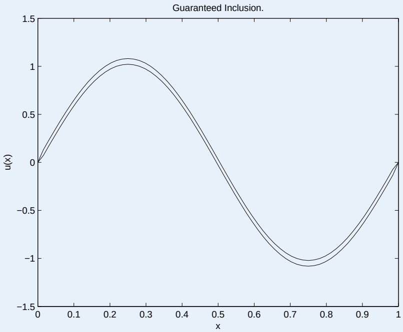

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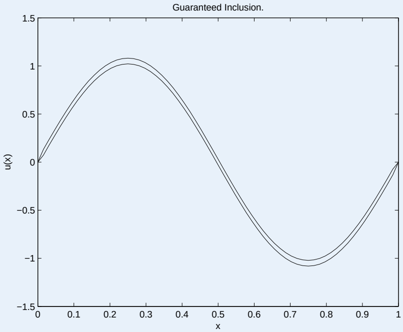

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Preliminary

- $L^2(\Omega)$: Lebesgue-measurable square-integrable functions with

$$(f,g) = \int_0^1 f(x)g(x)dx \quad \text{and} \quad \|f\|_{L^2} = \sqrt{(f,f)}, \quad (f,g \in L^2).$$

- $H^m(\Omega)$: L^2-Sobolev space of order m
- With $(\nabla u, \nabla v)$ and $\|u\|_{H_0^1} = \|\nabla u\|_{L^2}$,

$$H_0^1(\Omega) = \{u \in H^1 : \mathrm{tr}(u) = 0 \ (\text{on } \partial\Omega)\}.$$

- $L^\infty(\Omega)$: Space of functions that are essentially bounded on Ω with the norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_{a \leqq x \leqq b} |u(x)|.$$

Preliminary

- Let X and Y be Banach spaces. The set of linear bounded operators denoted by $\mathcal{L}(X,Y)$ with

$$\|L\|_{\mathcal{L}(X,Y)} = \sup_{u \in X \setminus \{0\}} \frac{\|Lu\|_Y}{\|u\|_X}, \quad (L \in \mathcal{L}(X,Y)).$$

  Here, $\|\cdot\|_X$ is the norm in X and $\|\cdot\|_Y$ is the norm in Y.
- Sobolev embedding theorem $(N = 2)$:
  (1) for $(k > l)$ the embedding $H^k(\Omega) \hookrightarrow H^l(\Omega)$ is compact and continuous,
  (2) let $q \in [1, \infty)$, the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact and continuous,
  (3) and $H_0^1(\Omega) \subset L^p(\Omega)$ for $p \geq 2$ with

$$\|v\|_{L^p} \leq C_{e,p} \|v\|_{H_0^1}, \quad v \in H_0^1(\Omega), \quad \left(\text{ex. } N = 2, \quad C_{e,p} = \frac{p\sqrt{2}}{8}\right).$$

Preliminary

Let $\Omega = \mathbf{R}^N, \quad (N = 1, 2)$ be the convex bounded polygonal domain.

Consider Dirichlet boundary value problems

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Let $f : H_0^1(\Omega) \to L^2(\Omega)$ be

$$f(u) = c_1 u + c_2 u^2 + c_3 u^3 + g.$$

$c_i \in L^\infty(\Omega), \ (i = 1, 2, 3), \ g \in L^2(\Omega).$

Preliminary

As an auxiliary problem, we consider Poisson equation.

For a given $g \in L^2$, the solution operator $\mathcal{K}$ corresponding to Poisson equation

$$\begin{cases} -\Delta u = g, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

leads the weak solution $u = \mathcal{K}g \in H^2 \cap H_0^1$.

- Define the solution operator $\mathcal{K} : L^2 \to H^2 \cap H_0^1$.
- Since the embedding $H^2 \hookrightarrow H^1$ is compact, $\mathcal{K} \in \mathcal{L}(L^2, H_0^1)$ and $f : H_0^1 \to L^2$, the composite operator

$$\mathcal{K}f : H_0^1 \to H_0^1$$

  becomes compact operator.

Established Algorithm

Outline of established method

Dirichlet boundary problems

$$(P)\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

are transformed into

$$(P) \iff u = \mathcal{K}f(u),\ (u \in H_0^1(\Omega)).$$

by using solution operator $\mathcal{K}$.
Define the operator $\mathcal{F} : H_0^1 \to H_0^1$ Newton-Kantorovich theorem is applied to a nonlinear operator equation:

$$\iff \mathcal{F}(u) = u - \mathcal{K}f(u) = 0.$$

Newton-Kantorovich Theorem

Let $\hat{u} \in H_0^1$ *and* $\mathcal{F} : H_0^1 \to H_0^1$ *be Fréchet differentiable. Assume that*

(1) *the Fréchet derivative* $\mathcal{F}'(u)$ *is nonsingular and satisfies*

$$\|\mathcal{F}'(\hat{u})^{-1}\mathcal{F}(\hat{u})\|_{H_0^1} \leqq \alpha$$

for a certain positive α.

(2) $\forall v, w \in B(\hat{u}, 2\alpha) = \{v \in H_0^1 : \|v - \hat{u}\|_{H_0^1} \leqq 2\alpha\} \subset H_0^1$,

$$\|\mathcal{F}'(\hat{u})^{-1}(\mathcal{F}'(v) - \mathcal{F}'(w))\|_{\mathcal{L}(H_0^1, H_0^1)} \leqq \omega \|v - w\|_{H_0^1}$$

is obtained for a certain positive ω.

If $\alpha\omega \leqq \frac{1}{2}$, *then there exists a solution* $u^* \in H_0^1$ *of* $\mathcal{F}(u) = 0$ *satisfying*

$$\|u^* - \hat{u}\|_{H_0^1} \leqq \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}.$$

Furthermore, the solution u^* *is unique in* $B(\hat{u}, \rho)$.

Newton-Kantorovich Theorem

For $u \in H_0^1$ we define an another linear operator $\mathcal{T} = \mathcal{T}(u) : H_0^1 \to H_0^1$ as

$$\mathcal{T}(u)v = \mathcal{K}(c_1 I + 2c_2 u + 3c_3 u^2)v.$$

Then the Fréchet derivative $\mathcal{F}'(u) : H_0^1 \to H_0^1$ is given as

$$\mathcal{F}'(u)v = (I - \mathcal{T}(u))v.$$

In order to valid the theorem, we compute three constants

$$\|\mathcal{F}'(\hat{u})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} = \|(I - \mathcal{T}(\hat{u}))^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq C_1,$$

$$\|\mathcal{F}(\hat{u})\|_{H_0^1} = \|\hat{u} - \mathcal{K}f(\hat{u})\|_{H_0^1} \leqq C_2$$

and

$$\|\mathcal{F}'(v) - \mathcal{F}'(w)\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq C_3\|v - w\|_{H_0^1}.$$

We set $\alpha = C_1 * C_2,\ \omega = C_1 * C_3$.

C_1 is tough to calculate

Theorem (Oishi 2000)

Let $\mathcal{T} : H_0^1 \to H_0^1$ be the compact operator, and $\mathcal{P}_n : H_0^1 \to X_n$ be the projection operator. Assume that there exist three constants K, L and M satisfying assumptions:

$$\|\mathcal{P}_n \mathcal{T}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq K,$$

$$\|\mathcal{T} - \mathcal{P}_n \mathcal{T}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq L$$

and

$$\|(I - \mathcal{P}_n \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq M.$$

If $(1 + MK)L < 1$, then the operator $(I - \mathcal{K})$ has the unique inverse operator which is estimated as

$$\|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq \frac{1 + MK}{1 - (1 + MK)L} =: C_1.$$

Inverse operator norm estimation algorithm

Compute rigorous upper bound of $\|(I-\mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)}$ by the following steps:

1 Compute $\|\hat{u}\|_{H_0^1}$ and calculate K and L by

$$K = C_{e,2}C_d(\|\hat{u}\|_{H_0^1}),$$

$$L = C_0 h C_d(\|\hat{u}\|_{H_0^1})$$

respectively.

2 Let D and G be $n \times n$ matrices whose i-j elements are given by

$$(\nabla\phi_j, \nabla\phi_i) \text{ and } (\nabla\phi_j, \nabla\phi_i) - (f'(\hat{u})\phi_j, \phi_i),$$

respectively. Let a lower triangular matrix L be the Cholesky decomposition of D, $D = LL^t$. If G is invertible, then set

$$M = \|L^t G^{-1} L\|_2.$$

When G is not invertible, stop with failure.

Inverse operator norm estimation algorithm

3 Check whether $(1+MK)L<1$ holds or not. If this holds, then the Fréchet derivative $\mathcal{F}'(u)$ is nonsingular. We have

$$\|\mathcal{F}'(\hat{u})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} = \|(I-\mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq \frac{1+MK}{1-(1+MK)L} =: C_1.$$

Otherwise, stop with failure.

Constant M takes much time to calculate.

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Omitting calculation of M

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$$\|\mathcal{P}_n \mathcal{T}\|_{\mathcal{L}(H_0^1, H_0^1)} \le K,$$

$$\|\mathcal{T} - \mathcal{P}_n \mathcal{T}\|_{\mathcal{L}(H_0^1, H_0^1)} \le L$$

and

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If $(1 + MK)L < 1$, *then the operator* $(I - \mathcal{K})$ *has the unique inverse operator which is estimated as*

$$\|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1, H_0^1)} \le \frac{1 + MK}{1 - (1 + MK)L} =: C_1.$$

Omitting calculation of M

Proposition (Takayasu)

Let $\mathcal{T} : H_0^1 \to H_0^1$ be the compact operator, and $\mathcal{P}_n : H_0^1 \to X_n$ be the projection operator. Assume that there exist two constants K and L satisfying assumptions:

$$\|\mathcal{P}_n \mathcal{T}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq K,$$

$$\|\mathcal{T} - \mathcal{P}_n \mathcal{T}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq L$$

If $K + L < 1$, then the operator $(I - \mathcal{K})$ has the unique inverse operator which is estimated as

$$\|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1, H_0^1)} \leq \frac{1}{1 - K - L} =: C_1.$$

Omitting calculation of M

Proof:

Since

$$u = (I - \mathcal{T})u + (\mathcal{T} - \mathcal{P}_n\mathcal{T})u + \mathcal{P}_n\mathcal{T}u,$$

then we have

$$\begin{aligned}\|u\|_{H_0^1} &\leqq \|(I-\mathcal{T})u\|_{H_0^1} + \|\mathcal{T} - \mathcal{P}_n\mathcal{T}\|_{\mathcal{L}(H_0^1,H_0^1)}\|u\|_{H_0^1} + \|\mathcal{P}_n\mathcal{T}u\|_{H_0^1} \\ &\leqq \|(I-\mathcal{T})u\|_{H_0^1} + L\|u\|_{H_0^1} + K\|u\|_{H_0^1}.\end{aligned}$$

Therefore, if $K + L < 1$,

$$\|u\|_{H_0^1} \leqq \frac{1}{1-K-L}\|(I-\mathcal{T})u\|_{H_0^1}.$$

From this inequality, if $(I - \mathcal{T})u = 0$, $u = 0$ follows. This implies the operator $(I - \mathcal{T}) : H_0^1(\Omega) \to H_0^1(\Omega)$ is injective.

Omitting calculation of M

Since the operator $(I - \mathcal{T}) : H_0^1(\Omega) \to H_0^1(\Omega)$ is of Fredholm type with the index 0, it is also surjective. Thus, $I - \mathcal{T} : H_0^1(\Omega) \to H_0^1(\Omega)$ is invertible $u = (I - \mathcal{T})^{-1} v$ for $v \in H_0^1(\Omega)$ and enjoys

$$\|(I - \mathcal{T})^{-1} v\|_{H_0^1} \leqq \frac{1}{1 - K - L} \|v\|_{H_0^1}.$$

It shows

$$\|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1, H_0^1)} \leqq \frac{1}{1 - K - L} = C_1.$$

□

A priori estimation for Linear case $(c_2 = c_3 = 0)$

Linear Problem

Let us consider linear Drichlet boundary value problems of the form

$$(P)\begin{cases} -\Delta u = c_1 u + f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

The same as the nonlinear case, by using the solutioin operator $\mathcal{K}$,

$$(P) \iff (I - \mathcal{K}c_1)u = \mathcal{K}f \iff (I - \mathcal{T})u = g.$$

where $(\mathcal{T} = \mathcal{K}c_1,\ g = \mathcal{K}f,\ f \in L^2(\Omega))$. If the operator $I - \mathcal{T}$ is invertible, there exits the unique solution with the error estimate

$$\begin{aligned} \|u - \hat{u}\|_{H_0^1} &= \|(I - \mathcal{T})^{-1}(I - \mathcal{T})(u - \hat{u})\|_{H_0^1} \\ &\le \|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(H_0^1, H_0^1)} \|g - (I - \mathcal{T})\hat{u}\|_{H_0^1} \\ &\le C_1 C_2. \end{aligned}$$

Numerical Examples

Numerical example 1

Example (Nearly singular problem)

Let $\Omega = (0,1)$, we consider

$$\begin{cases} -u'' = cu + 1 & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{ex1}$$

where $c = \pm 5, \pm 10$. Note that if $c = \pi^2 = 9.86...$ then this equation has no unique solution.

Numerical example 1

Verification Results of L^2-norm

Grid Points: 2^x	$c=-10$	$c=-5$	$c=5$	$c=10$
5	5.29×10^{-4}	1.06×10^{-3}	1.56×10^{-3}	$1.38\times10^{+0}$
6	1.35×10^{-4}	2.63×10^{-4}	3.89×10^{-4}	2.76×10^{-1}
7	3.40×10^{-5}	6.58×10^{-5}	9.72×10^{-5}	6.55×10^{-2}
8	8.54×10^{-6}	1.64×10^{-5}	2.43×10^{-5}	1.62×10^{-2}
9	2.14×10^{-6}	4.11×10^{-6}	6.07×10^{-6}	4.03×10^{-3}
10	5.36×10^{-7}	1.03×10^{-6}	1.52×10^{-6}	1.01×10^{-3}
11	1.34×10^{-7}	2.57×10^{-7}	3.80×10^{-7}	2.52×10^{-4}
12	3.35×10^{-8}	6.43×10^{-8}	9.52×10^{-8}	6.32×10^{-5}
13	-	1.64×10^{-8}	2.47×10^{-8}	-
14	-	4.85×10^{-9}	9.00×10^{-9}	-
Value of K	1.76	0.88	0.88	1.76

Numerical example 1

Execution Time ($c = 5$).

Grid Points: 2^x	Previous (s)	New (s)
5	0.061	0.037
6	0.092	0.043
7	0.151	0.048
8	0.279	0.051
9	0.704	0.055
10	2.279	0.063
11	10.42	0.091
12	44.62	0.136

CPU: 1.86 GHz Intel Core 2 Duo,
MATLAB 2009a with a toolbox for verified computations, INTLAB by Prof. Rump.

Numerical example 2

Example

$$\begin{cases} -u'' = u^3 - \cos 2\pi x & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

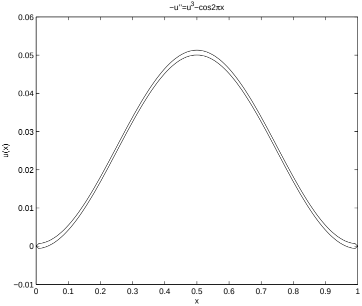

Figure: Guaranteed Inclusion of the Exact solution, $(n = 128)$.

Numerical example 2

Verification Results of H_0^1-norm

Grid Points: 2^x	Guaranteed Error: E_x	Ratio: $\frac{E_{x-1}}{E_x}$
5	4.99×10^{-3}	2.00
6	2.49×10^{-3}	2.00
7	1.25×10^{-3}	2.00
8	6.23×10^{-4}	2.00
9	3.12×10^{-4}	2.00
10	1.56×10^{-4}	2.00
11	7.79×10^{-5}	2.00
12	3.90×10^{-5}	2.00
13	1.95×10^{-5}	2.00
14	9.74×10^{-6}	2.00

Numerical example 2

The ratio of computational costs.

Grid Points: 2^x	Approximate (t_1)	Previous $(/t_1)$	New $(/t_1)$
5	8.86×10^{-4}	36.5	17.9
6	1.12×10^{-3}	43.8	15.1
7	1.08×10^{-3}	77.7	16.3
8	1.39×10^{-3}	118	12.7
9	2.03×10^{-3}	182	9.87
10	4.52×10^{-3}	216	7.11
11	1.86×10^{-2}	172	4.07
12	2.63×10^{-2}	423	3.26

t_1 is an approximate time.

CPU: 2.7 GHz, Quad-Core AMD Opteron(tm) Processor 8384,
MATLAB 2009a with a toolbox for verified computations, INTLAB.

Numerical example 3

Example

Let $\Omega = (0,1) \times (0,1)$,

$$\begin{cases} -\Delta u = u^2 + 10, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

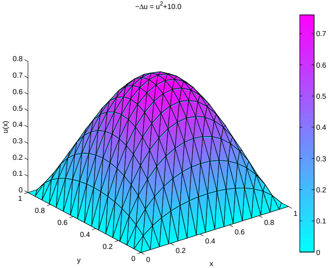

Figure: Shape of an approximate solution, (mesh size $1/16$).

Numerical example 3

Verification Results of H_0^1-norm

Grid Points: 2^x	Guaranteed Error: E_x	Ratio: $\frac{E_{x-1}}{E_x}$
5	2.08×10^{-1}	2.14
6	1.01×10^{-1}	2.06
7	4.98×10^{-2}	2.03
8	2.47×10^{-2}	2.01
9	1.23×10^{-2}	2.01
10	6.15×10^{-3}	2.00

Numerical example 3

The ratio of computational costs.

Mesh size: $1/2^x$	Approximate (t_1)	Previous ($/t_1$)	New ($/t_1$)
5	0.09	2.47	1.06
6	0.45	2.55	1.03
7	3.01	3.28	1.02
8	39.64	-	1.01
9	429.4	-	1.01

t_1 is an approximate time.

CPU: 2.7 GHz, Quad-Core AMD Opteron(tm) Processor 8384,
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Summery

Conclusion

- Established method requires much time to compute M.
- Omitting M in case of an condition.
- For easy problem, it is possible to calculate with high speed.
- Ratio is 2 - 10 times more than approximation.

Future Works

- How do I solve for difficult problem (In case of $K > 1$) ?
- We use a posteriori estimate?

Thank You for your kind Attention!