Interval Estimate of Structural Reliability

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Various sources of uncertainty

- Aleatory uncertainty
- Epistemic uncertainty
  - Model uncertainty
  - Statistical uncertainty (distribution form, distribution parameters, correlation, etc.)
- Reliability analysis under parameter uncertainty
  - Bayesian approach
  - Confidence interval approach

Bayesian approach

The basic reliability problem:

$$p_f = P(G(\mathbf{X}) \leq 0) = \int \cdots \int_{G(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}.$$

$\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)^T$ is the basic random variables. Let θ denotes the unknown parameters of $f_{\mathbf{x}}(\mathbf{x})$. θ are modeled as (Bayesian) random variables.

$$p_f(\theta) = P(G(\mathbf{X}, \theta) \leq 0) = \int \cdots \int_{G(\mathbf{X}, \theta) \leq 0} f_{\mathbf{X}|\theta}(\mathbf{x}|\theta) d\mathbf{x}.$$

We can estimate

- expected value of $p_f(\theta)$, or
- distribution of p_f, and obtain a confidence interval on p_f

Bayesian approach

- The mean (or other point estimate) of p_f does not fully characterise the epistemic uncertainty in the failure probability.
- Interval estimate of failure probability can provide useful information about the variability in reliability/risk
  - Subjective judgement is need to estimate $f_{\theta}(\theta)$
  - Computational cost

Interval approach and Probability Box

- The unknown parameter θ are modeled by interval bounds constructed from confidence intervals ($\theta \in \Theta$)
- Reliability analysis needs to consider *families* of distributions
- Bounding the unknown distributions (Dempster-Shafer evidence theory, random set, probability boxes)
- Lower and upper bounds on failure probability are computed
- computation procedure: a combination of interval analysis and Cartesian product method

Probability boxes

Let $F(x)$ denote the CDF for the random variable X. For every x, an interval generally can be readily found to bound the possible values of $F(x)$. i.e.

$$\underline{F}(x) \leq F(x) \leq \overline{F}(x), \text{ for } \theta \in \Theta$$

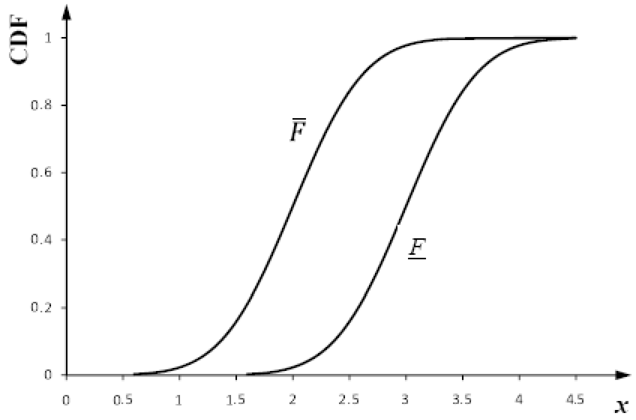

Figure 1: A probability box defined by a normal distribution with a mean of [2, 3] and a standard deviation of 0.5.

Interval Monte Carlo method

In Monte Carlo simulation, p_f is approximated as

$$p_f \approx \frac{1}{N} \sum_{j=1}^{N} I[G(\hat{\mathbf{x}}_j) \leq 0]$$

When **X** is a probability box with $\theta \in \Theta$, the limit state function becomes a function of θ. If the min and max of G can be computed

$$\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq G(\hat{\mathbf{x}}_j,\ \theta) \leq \text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)), \quad \text{for } \theta \in \Theta,$$

then

$$I[\text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0] \leq I[G(\hat{\mathbf{x}}_j,\ \theta) \leq 0] \leq I[\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0].$$

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then

$$I[\text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0] \leq I[G(\hat{\mathbf{x}}_j,\ \theta) \leq 0] \leq I[\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0].$$

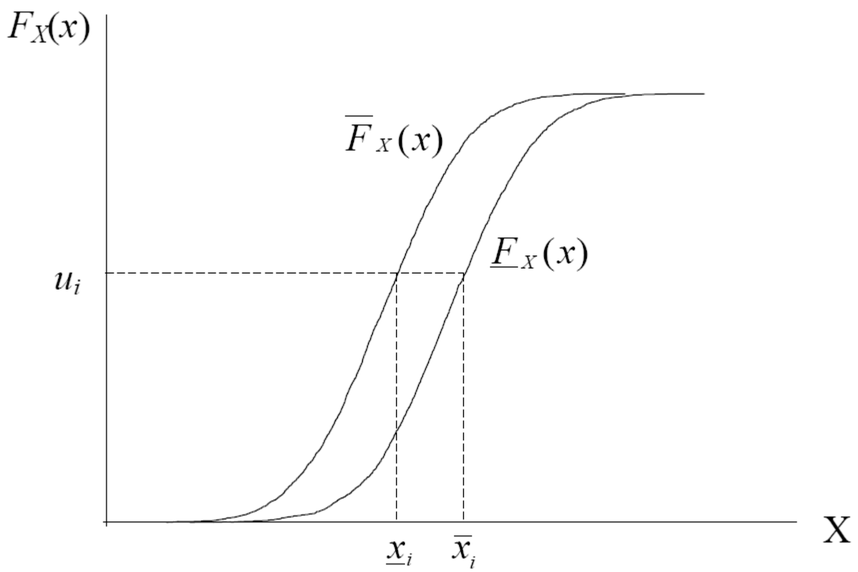
An interval estimate of p_f can be computed as

$$\underline{p}_f \approx \frac{1}{N} \sum_{j=1}^{N} I[\text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0],$$

$$\overline{p}_f \approx \frac{1}{N} \sum_{j=1}^{N} I[\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0], \quad \text{for } \theta \in \Theta.$$

Computational aspect

Random generation of intervals in accordance with the prescribed probability box–inverse transform method



$$\underline{x}_i = \overline{F}^{-1}(u_i);\ \overline{x}_i = \underline{F}^{-1}(u_i).$$

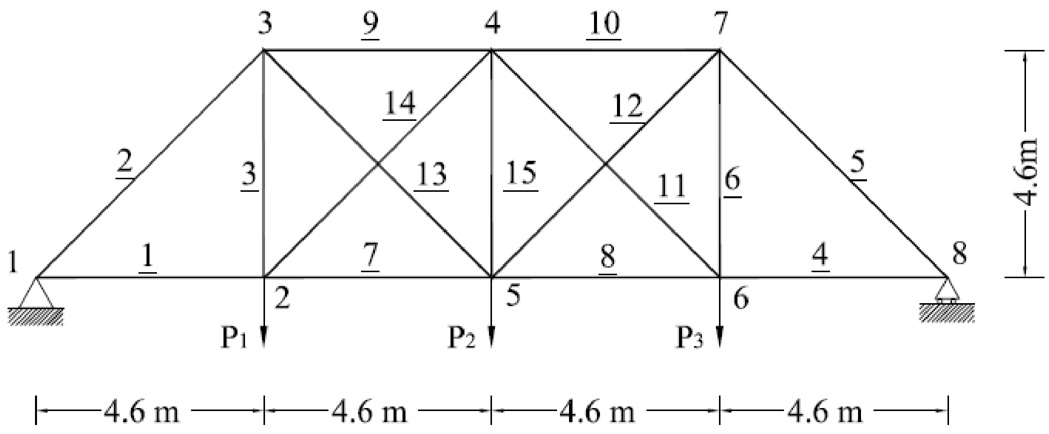
Computing the ranges of structure responses-interval FEM

- Computing the range of structure responses when the simulated basic variables vary in intervals
- Interval FEM
  - Optimization method
  - Interval analysis method
  - Perturbation method
  - Sensitivity analysis method

Interval Finite Element Method

- Interval arithmetic-based;
- Factorization of interval parameters out of element stiffness matrix;
- Element-by-element technique for element assembly;
- Penalty method or Lagrange multiplier method for imposition of constrains;
- Application and enhancement of the interval fixed point iteration for solution of the interval structural equations;
- Special algorithm for recovering stress/nodal forces.
- Removed most sources of overestimation.

Example 1



Serviceability criteria:
mid span displacement ≤ 7.5cm
Point estimate of P_f 0.14%

TABLE 1: Sample statistics for the basic random variables (truss in Fig. 4)

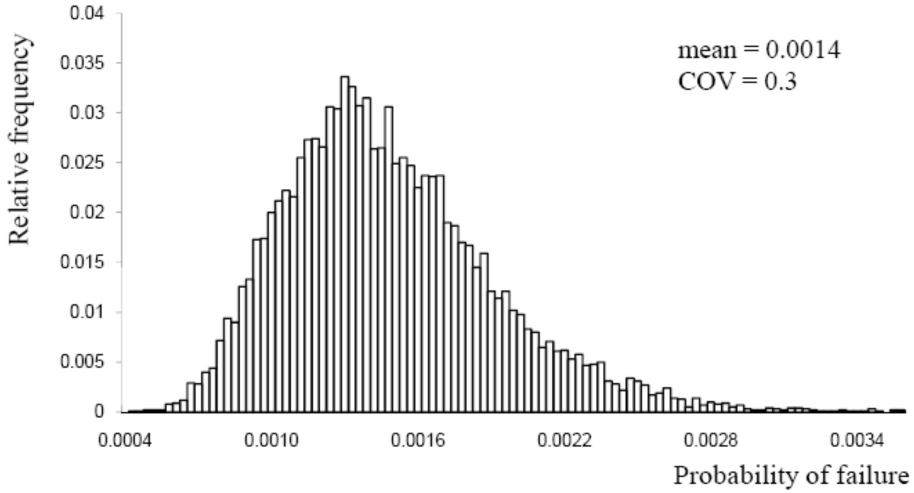
variables	sample mean	sample standard deviation	Num. samples
$A_1 - A_6$ (cm^2)	10.32	0.516	30
$A_7 - A_{15}$ (cm^2)	6.45	0.323	30
Ln P_1	4.483	0.09975	20
Ln P_2	5.582	0.09975	20
Ln P_3	4.483	0.09975	20

Unit of P: kN

All 18 random variables are mutually statistically independent.

Case 1- unknown means for the cross-sectional area

Bayesian approach (a uniform prior is assumed)



Population mean:
N (μ*, σ*/√n)

* denotes sample statistics.

Figure 5: Relative frequency histogram for p_f obtained from Bayesian approach, Case 1 (truss in Fig. 4).

Table 2: Bayesian confidence intervals for p_f, Case 1 (truss in Fig. 4).

Confidence	90%	95%	99%
p_f	[0.08%, 0.20%]	[0.063%, 0.22%]	[0.014%, 0.27%]

Case 1- unknown means for the cross-sectional area

Interval approach

Table 3: Interval estimates for p_f obtained from interval Monte Carlo simulations, Case 1 (truss in Fig. 4).

Variables	confidence level for the mean cross-sectional area 90%	95%	99%
$A_1 - A_6$ (cm^2)	[10.17, 10.48]	[10.14, 10.51]	[10.08, 10.56]
$A_7 - A_{15}$ (cm^2)	[6.35, 6.55]	[6.34, 6.57]	[6.30, 6.60]
p_f	[0.062%, 0.26%]	[0.048%, 0.30%]	[0.031%, 0.43%]

Bayesian approach

Table 2: Bayesian confidence intervals for p_f, Case 1 (truss in Fig. 4).

Confidence	90%	95%	99%
p_f	[0.08%, 0.20%]	[0.063%, 0.22%]	[0.014%, 0.27%]

Case 2- unknown means for Ln *P*

Bayesian approach (a uniform prior is assumed)

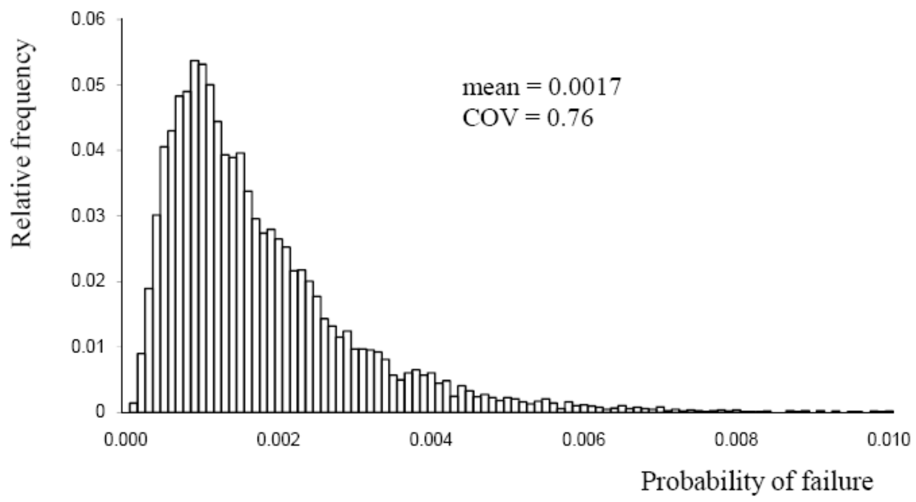


Figure 6: Relative frequency histogram for p_f obtained from Bayesian approach, Case 2 (truss in Fig. 4).

The failure probability is between 0.017% and 0.34% with 90% confidence.

Case 2- unknown means for Ln *P*

Interval approach

Table 4: Interval estimates for p_f obtained from interval Monte Carlo simulations, Case 2 (truss in Fig. 4).

Variables	confidence level for λ_i		
	90%	95%	99%
λ_1, λ_3	[4.4465, 4.5199]	[4.4395, 4.5269]	[4.4258, 4.5407]
λ_2	[5.5452, 5.6186]	[5.5381, 5.6256]	[5.5244, 5.6393]
p_f	[0.025%, 0.67%]	[0.019%, 0.72%]	[0.01%, 1.18%]

Comparison of Bayesian approach and Interval approach

- Both approaches can obtain interval estimates on p_f
- Both indicate that mean values of Ln P and member cross-sectional areas are important parameters.
- Additional data, particularly for loads, should be collected if the epistemic uncertainty in the failure probability is to be reduced.

Comparison of Bayesian approach and Interval approach

- Bayesian approach
  - Prior distribution on the unknown parameters are assumed and incorporated.
  - Bayesian confidence interval on p_f has statistical implications.
- Interval approach
  - Only uses the information of the confidence bounds on unknown parameters.
  - Requires less subjective information than Bayesian approach.
  - No direct statistical implication in the interval failure probability.

Example 2

Drift criteria: $\leq H/500$ (21 mm)

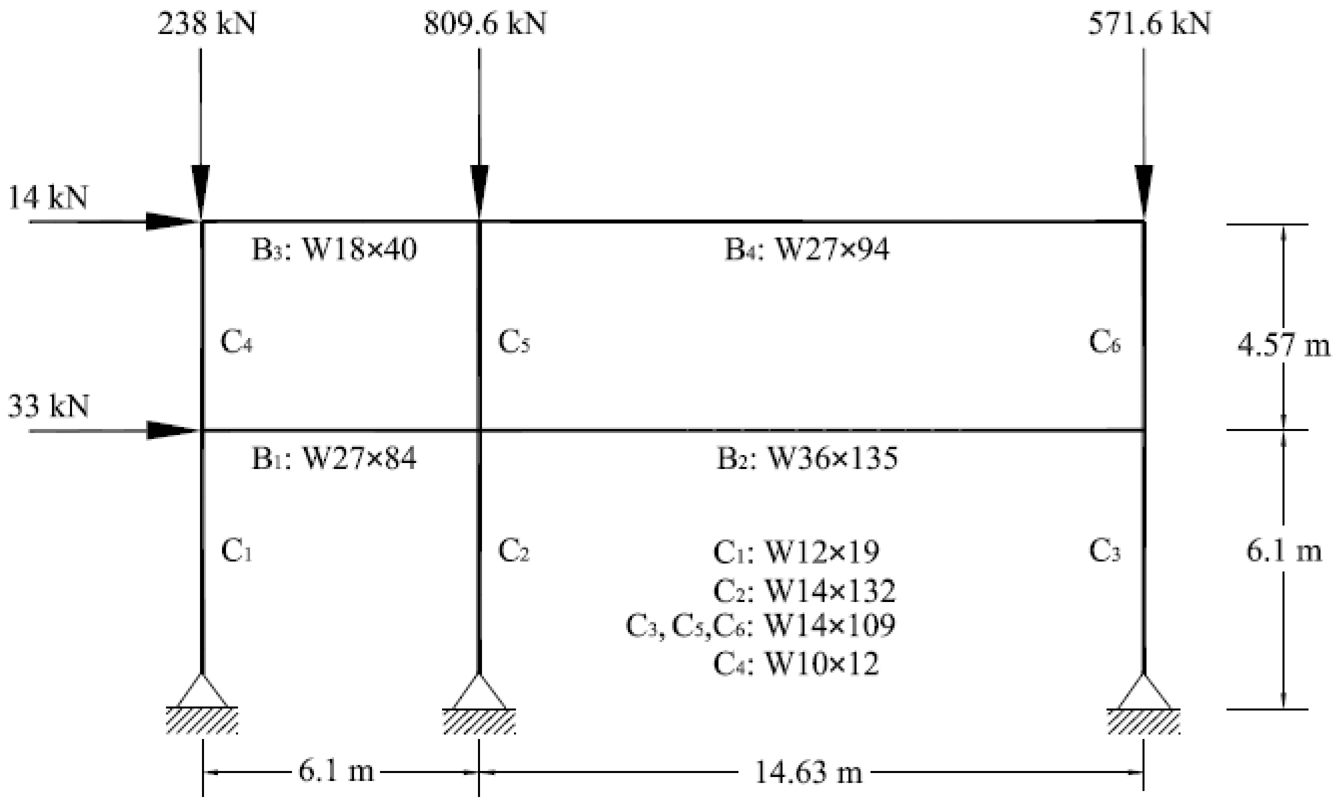


Figure 7: Two-bay two-story frame.

Example 2

Table 5: Random sectional properties for the frame of Fig. 7.

Member	μ_I (cm^4)	σ_I (cm^4)	μ_A (cm^2)	σ_A (cm^2)
C_1	[5327.2, 5494.9]	270.55	[35.4, 36.5]	1.80
C_2	[62696.6, 64670.2]	3184.2	[246.4, 254.2]	12.5
C_4	[2204.6, 2274.0]	111.97	[22.5, 23.2]	1.14
C_3 C_5 C_6	[50813.0, 52412.4]	2580.6	[203.3, 209.7]	10.32
B_1	[116787.9, 120464.1]	5931.3	[157.5, 162.5]	8.0
B_2	[319629.9, 329691.1]	16233.0	[252.2, 260.1]	12.81
B_3	[25078.7, 25868.1]	1273.7	[75.0, 77.3]	3.81
B_4	[133998.7, 138216.6]	6805.4	[176.0, 181.5]	8.94

μ: mean; σ: standard deviation.

Perfect correlation assumed between col-col, beam-beam; no correlation between col-beam.

Example 2

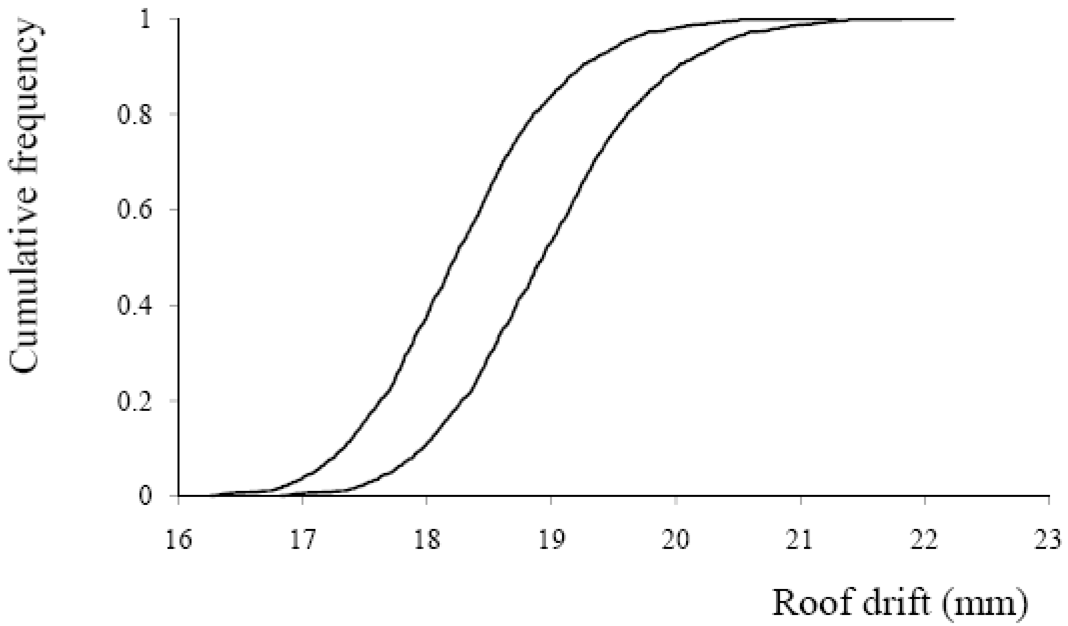


Figure 8: Bounds of the cumulative frequency distribution of the roof drift (frame in Fig. 7).

p_f = [0.11%, 1.29%] (10,000 interval MC)

Conclusion

- An interval Monte Carlo method has been developed for reliability assessment under parameter uncertainty.
- Aleatory and epistemic uncertainty are propagated separately through reliability analysis.
- Interval estimate of P_f can provide a statement of confidence in the results of reliability assessment.
- Interval P_f from the proposed method tends to be wider than that from Bayesian approach.

The presentation is based on the journal paper:

Zhang, H, Mullen, RL, and Muhanna RL (2010). “Interval Monte Carlo methods for structural reliability.” Structural Safety, in press, available online 2 February 2010

Thank you!