RELIABLE DYNAMIC ANALYSIS OF AN UNCERTAIN SHEAR BEAM

Mehdi Modares, Ignacio Olabarri, and Robert L. Mullen

DEPARTMENT OF CIVIL, ARCHITECTURAL AND EVIRONMENTAL ENGINEERING

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Shear Beams

In structural engineering, shear beams and their dynamic behavior play an important role in

- Modeling,
- Analysis, and
- Design

of various types of structures subjected to a system of dynamic loads such as wind or earthquake excitations.

Shear Beam Significance

- The dynamic properties of the system are sufficiently simple.
- Therefore, it serves as a convenient representation for illustrating various fundamental features of earthquake response of structures.

Shear Beam Significance

- It introduces the dynamic analysis of continuous systems with significant shear effects.
- Moreover, the uniform Shear Beam serves as an approximate model for earthquake and wind response of very important types of structures: moderately tall and regular buildings.

Shear Beam (Cont.)

In current procedures for dynamic analysis of shear beams, the existence of uncertainty in either system's mechanical properties and/or the characteristics of forcing function is not considered.

Uncertainties

Uncertainties can be attributed to:

- Physical imperfections,
- Modeling inaccuracies, and
- Load-system interaction complexities.

For reliable design, the presence of uncertainties must be included in analysis procedures.

Interval Uncertainty

- The interval representation is one method to quantify the uncertainty present in a physical system.
- In this representation, the uncertain parameter varies within the interval defined by extreme values.

Objective

To develop a method for dynamic analysis of a shear beam with properties of structure and load expressed as interval quantities.

Presentation Outline

- Review of Deterministic Continuous Dynamic Analysis for Shear Beams
- Interval Concepts
- Introduce Interval Continuous dynamic Analysis for Shear Beams
- Numerical Example
- Conclusions

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Shear Beam

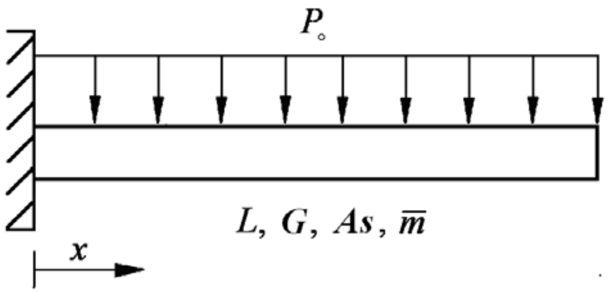
- The shear beam theory was developed by Ukrainian/Russian-born scientist Stephen Timoshenko in the beginning of the 20th century.

- Suitable for describing the behavior of short beams, sandwich composite beams and moderately tall buildings.

- The model takes into account shear deformation and rotational inertia effects.

Dynamic Deterministic Analysis

Considering a cantilever shear beam subjected to a suddenly applied uniformly distributed load:



The partial differential equation of motion is:

$$\frac{\partial}{\partial x}\left(GA_s \frac{\partial u(x,t)}{\partial x}\right) + P_\circ = \overline{m}\,\frac{\partial^2 u(x,t)}{\partial t^2}$$

Solution
Free Vibration

Considering the free vibration and assuming a harmonic function:

$$u(x,t) = \varphi(x)e^{i\omega t}$$

Substitute in the equation of motion, the linear eigenvalue problem:

$$-\frac{\partial}{\partial x}\left(GA_s \frac{d\varphi(x)}{dx}\right) = \overline{m}\,\omega^2 \varphi(x)$$

Considering:

$$b^2 = \frac{\overline{m}\,\omega^2}{GA_s}$$

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Solution
Free Vibration

Considering a solution:

$$\varphi(x) = A\sin(bx) + B\cos(bx)$$

Applying boundary conditions for simply-supported beam:

$$\varphi(0) = \varphi'(L) = 0$$

The non-trivial solution to the characteristic equation can be obtained.

Solution
Free Vibration

Natural circular frequencies (Eigenvalue):

$$\omega_n = (2n-1)\frac{\pi}{2}\sqrt{\frac{GA_s}{\overline{m}L^2}}$$

Mass-orthonormalized mode shape (Eigenfunction):

$$\varphi_n(x) = \sqrt{\frac{2}{\overline{m}L}}\sin\left((2n-1)\frac{\pi x}{2L}\right)$$

Normalized by:

$$\int_0^L \varphi_n(x)\overline{m}\,\varphi_n(x) = 1$$

Solution
Forced Vibration

The solution for the forced vibration may be expressed:

$$u(x,t) = \sum_{n=1}^{\infty} y_n(t)\hat{\varphi}_n(x)$$

where, y_n is the modal coordinate.

Substituting in equation of motion, decoupling by:

- Premultiplying by each mode shape
- Integrating over the domain
- Invoking orthogonality
- Adding modal damping ratio

Decoupled System

The modal equation of motion is:

$$\ddot{y}_n(t) + 2\zeta_n\omega_n\dot{y}_n(t) + \omega_n^2 y_n(t) = \int_0^L \hat{\varphi}_n(x) P_\circ dx$$

or

$$\ddot{y}_n(t) + 2\zeta_n\omega_n\dot{y}_n(t) + \omega_n^2 y_n(t) = \frac{2P_o}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos(\frac{(2n-1)\pi L}{2L})\right)$$

Considering $\Gamma_n = \frac{2P_o}{\sqrt{\overline{m}}(2n-1)\pi}$ as the modal participation factor,

Updated Modal Coordinate

Defining an updated modal coordinate:

$$d_n(t) = \frac{y_n(t)}{\Gamma_n}$$

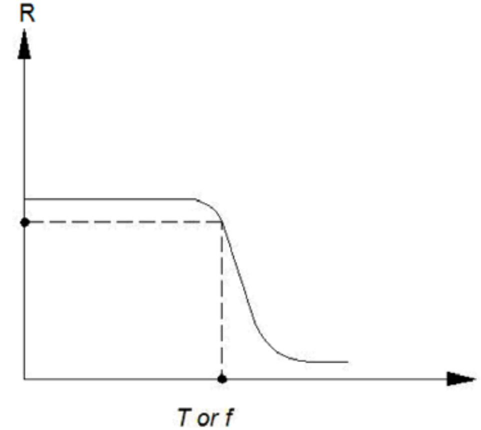
The updated modal equation:

$$\ddot{d}_n(t) + 2\zeta_n \omega_n \dot{d}_n(t) + \omega_n^2 d_n(t) = 1$$

Maximum Modal Coordinate

The maximum modal coordinate is obtained using response spectrum of each mode for a given ω_n and ζ_n .

Response Spectrum: Function of maximum dynamic amplification response versus the natural frequencies for an assumed damping ratio.



Maximum Modal Response

The maximum modal displacement response is the product of:

- Maximum modal coordinate
- Modal participation factor
- Mode shape

$$u_{n,\max} = (d_{n,\max})(\Gamma_n)(\phi_n(x))$$

or:

$$u_{n,\max} = (d_{n,\max})\frac{8\sqrt{2}P_o}{\overline{m}\sqrt{(2n-1)\pi}}\sin^3\left((2n-1)\frac{\pi x}{4L}\right)\cos\left((2n-1)\frac{\pi x}{4L}\right)$$

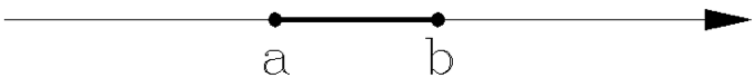
Dynamic Deterministic Analysis

Finally, the total displacement response is obtained using superposition (Square Root of Sum of Squares) of modal maxima:

$$u_{\max} = \sqrt{\sum_{n=1}^{N} (u_{n,\max})^2}$$

No consideration of uncertainty in the scheme.

Interval Variable



$$\tilde{x} = [a, b]$$

A real interval is a set of the form:

$$\tilde{Z} = [z^l, z^u] = \{z \in \Re \mid z^l \le z \le z^u\}$$

Archimedes (287-212 B.C.)

$$(3\frac{10}{71} < \pi < 3\frac{1}{7})$$

Interval Dynamic Analysis

Considering a beam with uncertain modulus of elasticity subjected to an uncertain load $\tilde{G} = [G_l, G_u]$, $\tilde{P}_o = [P_o^l, P_o^u]$.

The partial differential equation of motion:

$$\frac{\partial}{\partial x}\left(\tilde{G}A_s \frac{\partial u(x,t)}{\partial x}\right) + \tilde{p}(x,t) = m(x)\frac{\partial^2 u}{\partial t^2}$$

Solution

Natural circular frequencies (Eigenvalue):

$$\tilde{\omega}_n = (2n-1)\frac{\pi}{2}\sqrt{\frac{\tilde{G}A_s}{\overline{m}L^2}}$$

Mass-orthonormalized mode shape (Eigenfunction):

$$\phi_n(x) = \sqrt{\frac{2}{\overline{m}L}}\sin\left((2n-1)\frac{\pi x}{2L}\right)$$

Frequencies' Monotonic Behavior

Re-writing interval natural circular frequencies:

$$\tilde{\omega}_n = (2n-1)\frac{\pi}{2}\left[\sqrt{\tilde{G}^L}, \sqrt{\tilde{G}^U}\right]\cdot\sqrt{\frac{A_s}{\overline{m}L^2}}$$

This leads to an evident realization of monotonic behavior of natural circular frequencies due to variation in stiffness in continuous dynamic systems.

Interval Eigenvalue Problem in Discrete Systems

Interval eigenvalue problem using the interval global stiffness matrix:

$$(\sum_{i=1}^{n}([l_i, u_i])[\overline{K}_i])\{\varphi\} = (\tilde{\omega}^2)[M]\{\varphi\}$$

Rayleigh quotient (ratio of quadratics):

$$R(x) = \frac{x^T A x}{x^T x}$$

Bounds on Natural Frequencies

The first eigenvalue – Minimum:

$$\tilde{\lambda}_1 = \min_{x \in R^n} \left[\frac{x^T (\sum_{i=1}^{n} ([l_i, u_i])[\overline{K}_i]) x}{x^T M x} \right] = \min_{x \in R^n} \left(\sum_{i=1}^{n} ([l_i, u_i]) \frac{x^T [\overline{K}_i] x}{x^T M x} \right)$$

The next eigenvalues – Maximin Characterization:

$$\tilde{\lambda}_k = \max \left[\min_{x.z_i = 0, i=1,\dots,k-1} \frac{x^T (\sum_{i=1}^{n} ([l_i, u_i])[\overline{K}_i]) x}{x^T M x} \right] = \max \left[\min_{x.z_i = 0, i=1,\dots,k-1} \sum_{i=1}^{n} ([l_i, u_i]) \frac{x^T [\overline{K}_i] x}{x^T M x} \right]$$

Bounding Deterministic Eigenvalue Problems

Solution to interval eigenvalue problem correspond to the maximum and minimum natural frequencies:

$$(\sum_{i=1}^{n}(u_i)[\overline{K}_i])\{\varphi\} = (\omega_{\max}^2)[M]\{\varphi\}$$

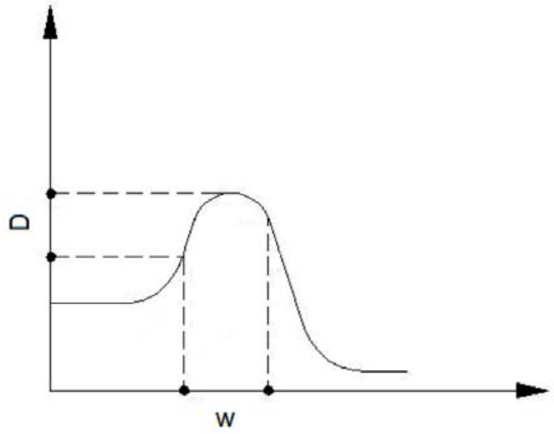
$$(\sum_{i=1}^{n}(l_i)[\overline{K}_i])\{\varphi\} = (\omega_{\min}^2)[M]\{\varphi\}$$

Two deterministic problems capable of bounding all natural frequencies of the interval system

(Modares and Mullen 2004)

Maximum Modal Coordinate

Having the interval natural frequency, the interval modal coordinate is determined using modal response spectrum as:



The maximum modal coordinate:

$$d_{n,\max} = \max(\partial_n)$$

Maximum Modal Participation Factor

The interval modal participation factor:

$$\tilde{\Gamma}_n = \frac{2\tilde{P}_o}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos(\frac{(2n-1)\pi L}{2L})\right)$$

The maximum modal participation factor:

$$\Gamma_{n,\max} = \frac{2P_o^u}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos(\frac{(2n-1)\pi L}{2L})\right)$$

Maximum Modal Response

The maximum modal displacement response is the product of:

- Maximum modal coordinate
- Modal participation factor
- Mode shape

$$u_{n,\max} = (d_{n,\max})(\Gamma_{n,\max})(\hat{\varphi}_n(x))$$

or:

$$u_{n,\max} = (d_{n,\max})\frac{8\sqrt{2}P_o^u}{\overline{m}\sqrt{(2n-1)\pi}}\sin^3\left((2n-1)\frac{\pi x}{4L}\right)\cos\left((2n-1)\frac{\pi x}{4L}\right)$$

Deterministic Dynamic Analysis

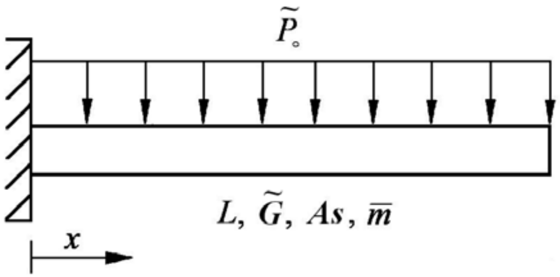
Finally, the total displacement response is obtained using superposition (Square Root of Sum of Squares) of modal maxima:

$$u_{\max} = \sqrt{\sum_{n=1}^{N} (u_{n,\max})^2}$$

Uncertainties are included in the response.

Numerical Example

Dynamic analysis of a continuous cantilever shear beam with interval uncertainty in the shear modulus and magnitude of uniformly distributed load.



Properties:

$L = 10\,ft$ $\quad I = 1530 in^4$ $\quad \overline{m} = 132 lb/g$ $\quad \zeta = 1\%$ $\quad \tilde{G} = ([0.9, 1.1])11200 ksi$

$A_s = 25.23 in^2$ $\quad \tilde{P}_\circ = [0.9, 1.1] P_\circ$

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Numerical Example

The problem is solved by:

- The present method
- Monte-Carlo simulation
  (Using bounded uniformly distributed random variables in 1000 simulations)

Results

Bounds on the fundamental natural frequency (first mode)

	Lower Bound *Present Method*	Lower Bound *Monte-Carlo Simulation*	Upper Bound *Monte-Carlo Simulation*	Upper Bound *Present Method*
ω_1	39.12162	39.12733	43.24335	43.25451

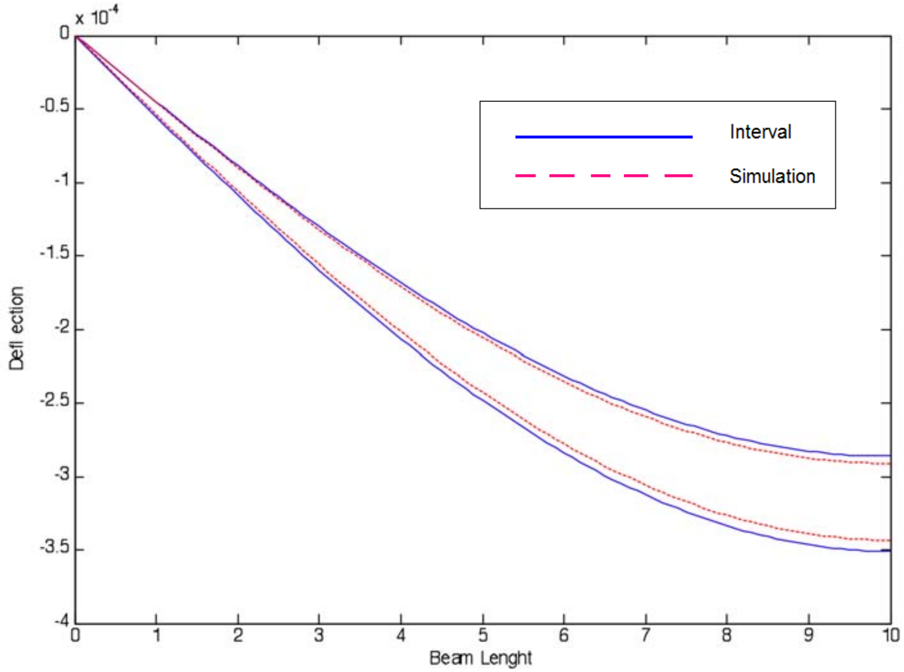
Numerical Example

The upper bounds of the maximum displacement response at the beam's free-end

	Upper Bound *Monte-Carlo Simulation*	Upper Bound *Present Method*
$\frac{u_{max}}{P_{\circ}}$	0.350714e-3	0.350816e-3

Results

The first three-mode beam deflection response:



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Sensitivity Analysis

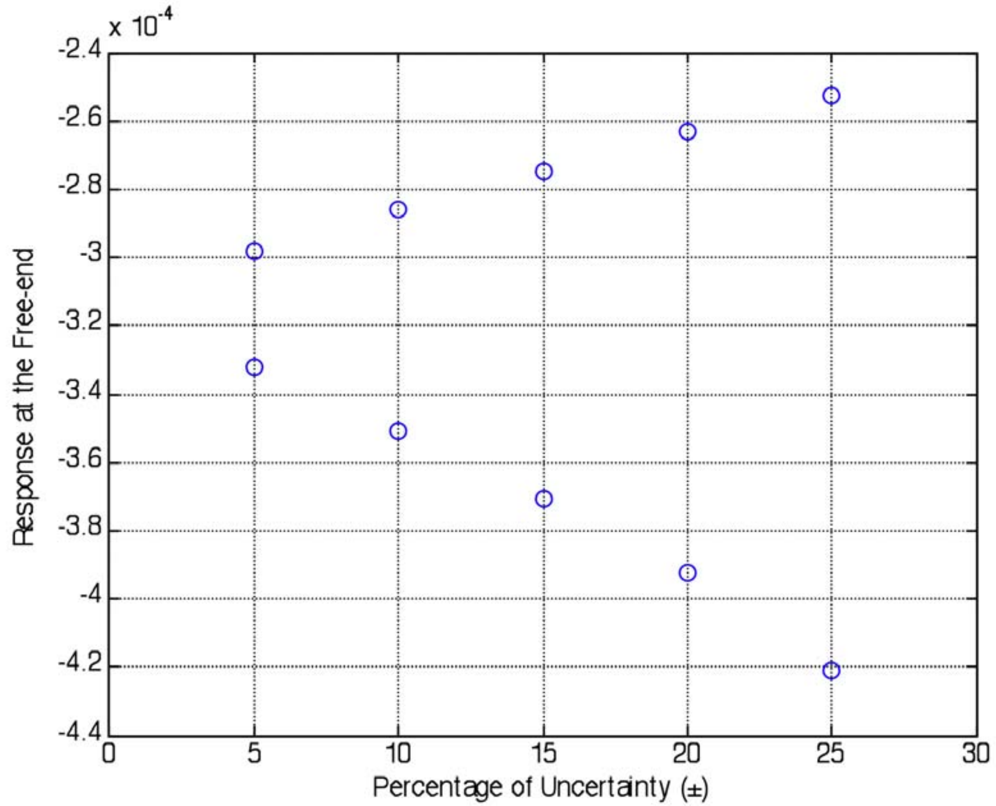
- Performed to investigate the robustness of the proposed method.

- The shear modulus of elasticity is varied as

$$\tilde{G}(n) = 1 + n \times [-0.1, 0.1] \times mid(\tilde{G})$$

$$n = 1,2,...5$$

Results



x 10^-4
-2.4
-2.6
-2.8
-3
-3.2
-3.4
-3.6
-3.8
-4
-4.2
-4.4
Response at the Free-end
0
5
10
15
20
25
30
Percentage of Uncertainty (±)

Results

The sensitivity analysis shows that:

- The growth in width of interval of dynamic response is linear.
- In the case of increasing uncertainty, the proposed methodology is robust.

Conclusions

- A new method for dynamic analysis of shear beam with uncertainty in the mechanical characteristics of the system as well as the properties of the load is developed.

- This computationally efficient method shows that implementation of interval analysis in a continuous dynamic system yields exact and robust results and preserves the problem's physics.

Conclusions

- The results show that obtaining bounds does not require expensive stochastic procedures such as Monte-Carlo simulations.

- The simplicity of the proposed method makes it attractive to introduce uncertainty in analysis of continuous dynamic systems.

Questions