Primary and Derived Variables with the Same Accuracy in Interval Finite Elements

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Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Overestimation in IFEM
- New Formulation
- Examples
- Conclusions

Introduction- Uncertainty

- Uncertainty is unavoidable in engineering system
  - Structural mechanics entails uncertainties in material, geometry and load parameters (aleatory-epistemic)
- Probabilistic approach is the traditional approach
  - Requires sufficient information to validate the probabilistic model
  - What if data is insufficient to justify a distribution?

Introduction- Uncertainty

Information

↓

Available Information

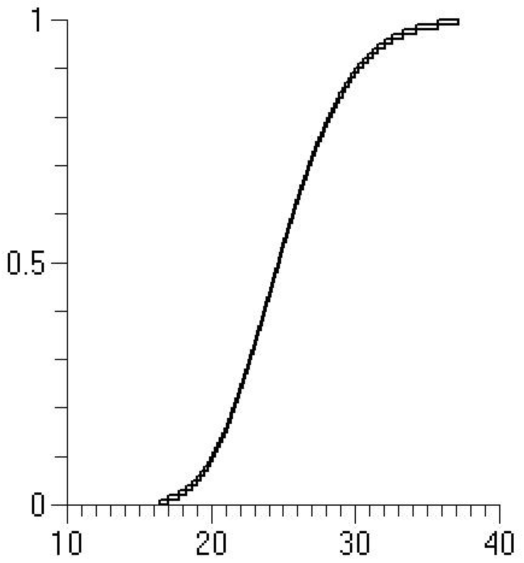
↓

Sufficient → Probability

Incomplete → Imprecise Probability

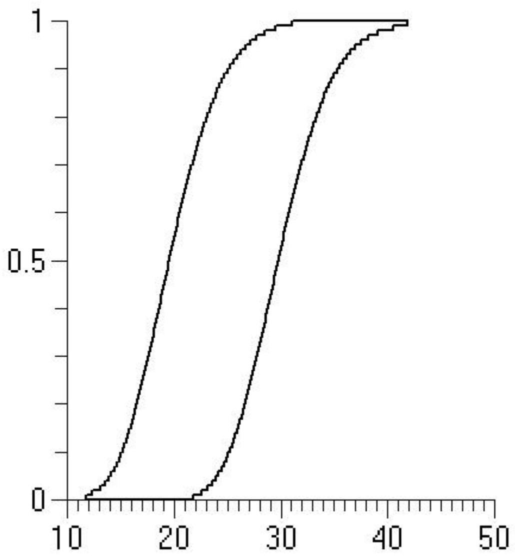
Introduction- Uncertainty

Lognormal



Probability

Lognormal with interval mean



Imprecise Probability

Tucker, W. T. and Ferson, S. , Probability bounds analysis in environmental risk assessments, Applied Biomathematics, 2003. Mean = [20, 30], Standard deviation = 4, truncated at 0.5th and 99.5th.

Introduction- Interval Approach

- Only range of information (tolerance) is available

$$t = t_0 \pm \delta$$

- Represents an uncertain quantity by giving a range of possible values

$$t = [t_0 - \delta,\ t_0 + \delta]$$

- How to define bounds on the possible ranges of uncertainty?
  - experimental data, measurements, statistical analysis, expert knowledge

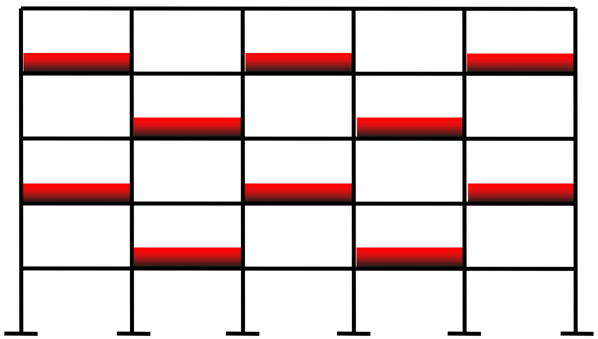
Introduction- Why Interval?

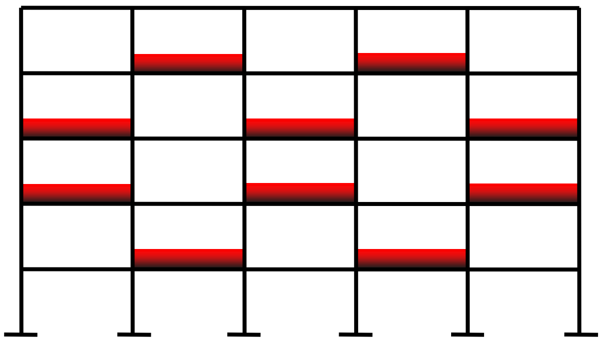
- Simple and elegant
- Conforms to practical tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set, imprecise probability)

- **Provides guaranteed enclosures**

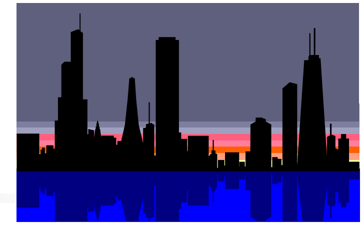
Examples – Load Uncertainty

- Four-bay forty-story frame

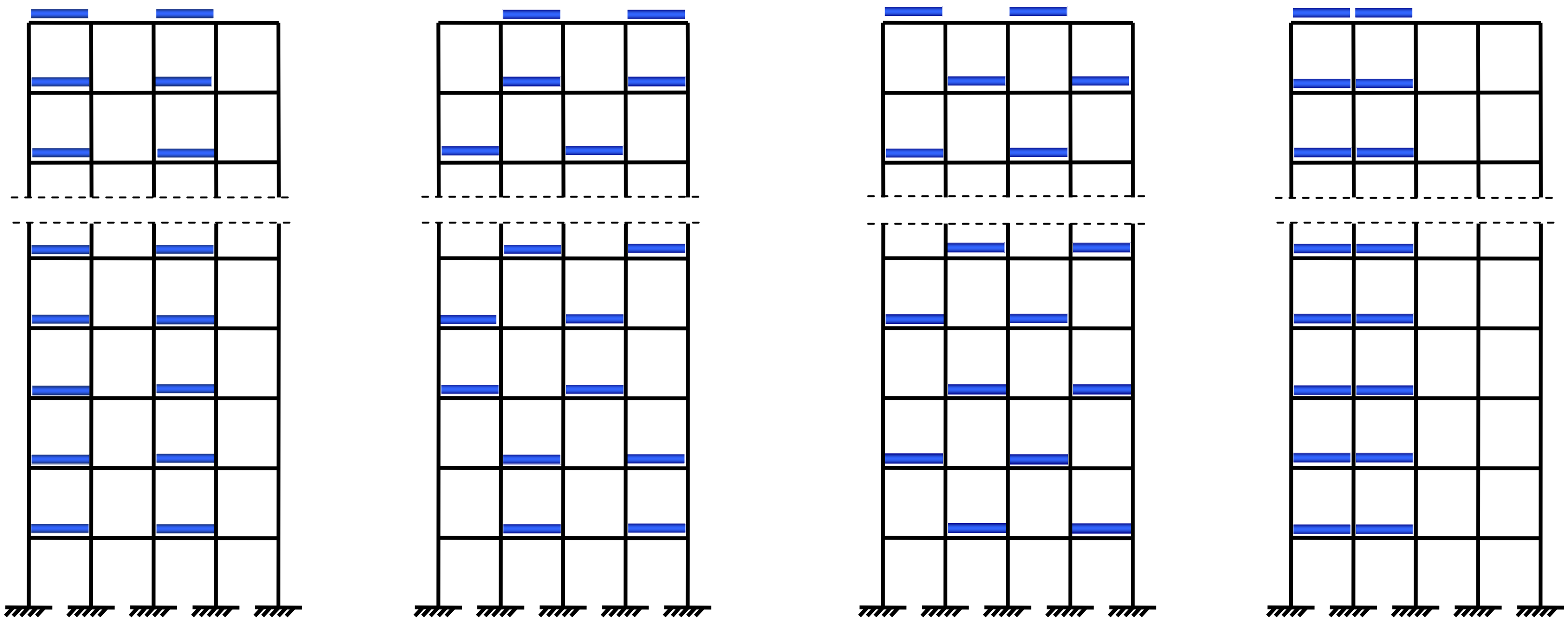




Examples – Load Uncertainty



- **Four-bay forty-story frame**



Loading A   Loading B   Loading C   Loading D

Examples – Load Uncertainty

- **Four-bay forty-story frame**

Total number of floor load patterns

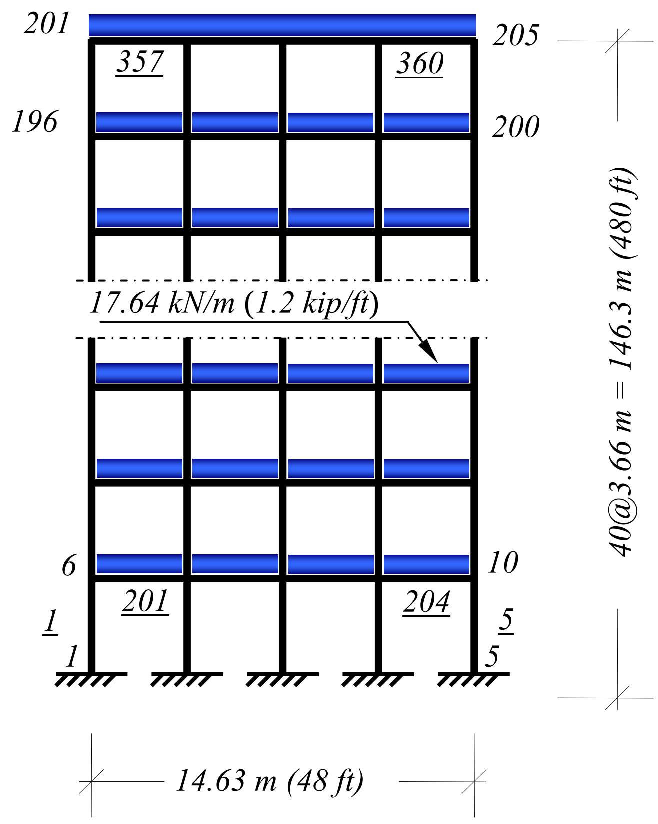
$2^{160} = 1.46 \times 10^{48}$

If one were able to calculate

10,000 *patterns / s*

there has not been sufficient time since the creation of the universe **(4-8)** billion years ? to solve all load patterns for this simple structure

Material *A36*, Beams *W24* x *55*, Columns *W14* x *398*



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Interval arithmetic

- Interval number represents a range of possible values within a closed set

$$\boldsymbol{x} \equiv [\underline{x}, \overline{x}] := \{x \in R \mid \underline{x} \le x \le \overline{x}\}$$

Properties of Interval Arithmetic

Let x, y and z be interval numbers

1. Commutative Law

$$x + y = y + x$$

$$xy = yx$$

2. Associative Law

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

3. ***Distributive Law does not always hold,*** *but*

$$x(y + z) \subseteq xy + xz$$

Sharp Results – Overestimation

- The ***DEPENDENCY*** problem arises when one or several variables occur more than once in an interval expression

  - $f(\mathbf{x}) = \mathbf{x} - \mathbf{x}$, $\mathbf{x} = [1, 2]$
  - $f(\mathbf{x}) = [1 - 2, 2 - 1] = [-1, 1] \neq 0$
  - ~~$f(\mathbf{x}, \mathbf{y}) = \{ f(x, y) = x - y \mid x \in \mathbf{x}, y \in \mathbf{y} \}$~~

  - $f(\mathbf{x}) = \mathbf{x}\,(1 - 1) \quad \Rightarrow \quad f(\mathbf{x}) = 0$
  - $f(\mathbf{x}) = \{ f(x) = x - x \mid x \in \mathbf{x} \}$

Sharp Results – Overestimation

- Let ***a, b, c*** and ***d*** be independent variables, each with interval [1, 3]

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ -\boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \quad A \times \boldsymbol{B} = \begin{pmatrix} [-2, 2] & [-2, 2] \\ [-2, 2] & [-2, 2] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B}_{phys} = \begin{pmatrix} \boldsymbol{b} & -\boldsymbol{b} \\ -\boldsymbol{b} & \boldsymbol{b} \end{pmatrix}, \quad A \times \boldsymbol{B}_{phys} = \begin{pmatrix} [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \\ [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B^{*}_{phys} = \boldsymbol{b} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A \times \boldsymbol{B}^{*}_{phys} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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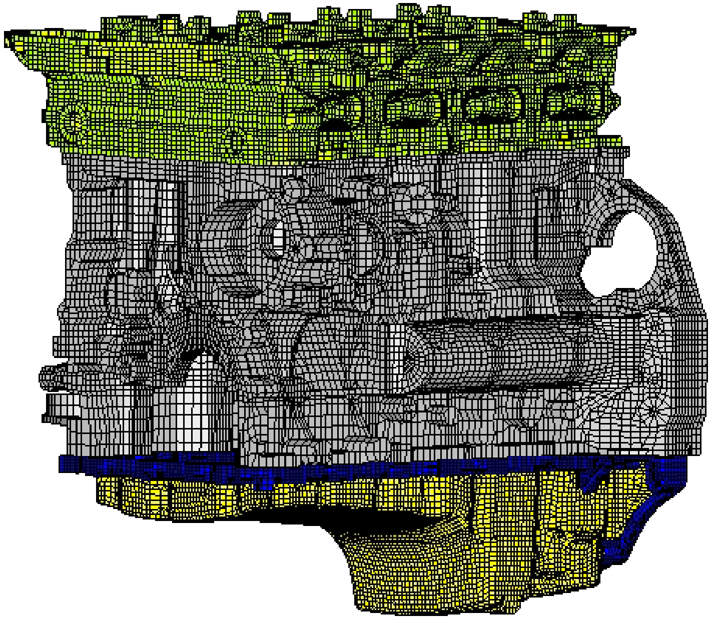
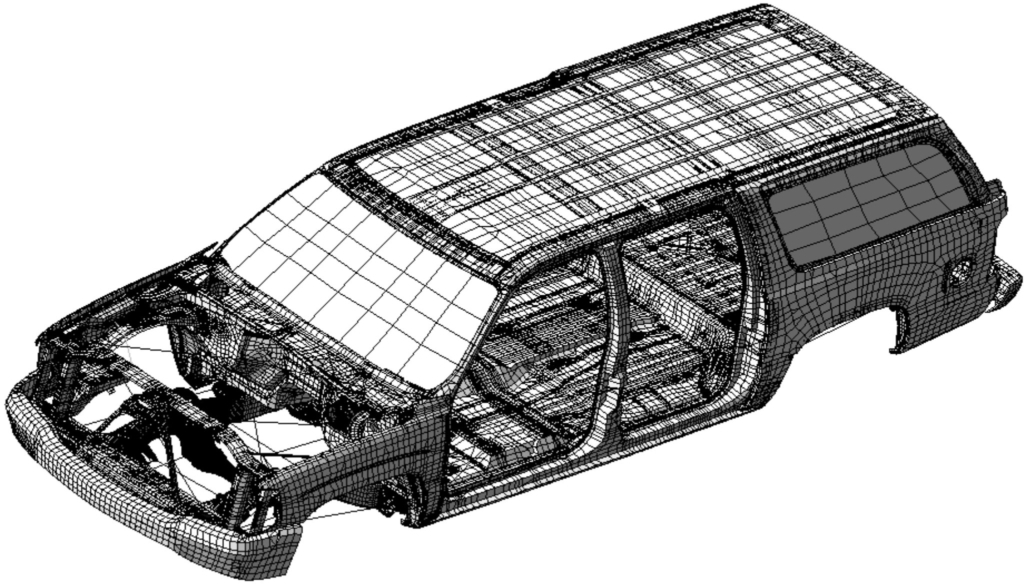
Finite Elements

Finite Element Methods (FEM) are numerical method that provide approximate solutions to differential equations (ODE and PDE)

Finite Elements





Finite Element Model (courtesy of Prof. Mourelatous)
500,000-1,000,000 equations

Finite Elements- Uncertainty& Errors

- Mathematical model (validation)
- Discretization of the mathematical model into a computational framework (verification)
- Parameter uncertainty (loading, material properties)
- Rounding errors

Interval Finite Elements (IFEM)

- Follows conventional FEM
- Loads, geometry and material property are expressed as interval quantities
- System response is a function of the interval variables and therefore varies in an interval
- Computing the exact response range is proven NP-hard
- The problem is to estimate the bounds on the unknown exact response range based on the bounds of the parameters

FEM- Inner-Bound Methods

- Combinatorial method (Muhanna and Mullen 1995, Rao and Berke 1997)
- Sensitivity analysis method (Pownuk 2004)
- Perturbation (Mc William 2000)
- Monte Carlo sampling method
- Need for alternative methods that achieve
  - Rigorousness – guaranteed enclosure
  - Accuracy – sharp enclosure
  - Scalability – large scale problem
  - Efficiency

IFEM- Enclosure

- Linear static finite element
  - Muhanna, Mullen, 1995, 1999, 2001,and Zhang 2004
  - Popova 2003, and Kramer 2004
  - Neumaier and Pownuk 2004
  - Corliss, Foley, and Kearfott 2004
- Heat Conduction
  - Pereira and Muhanna 2004
- Dynamic
  - Dessombz, 2000
- Free vibration-Buckling
  - Modares, Mullen 2004, and Bellini and Muhanna 2005

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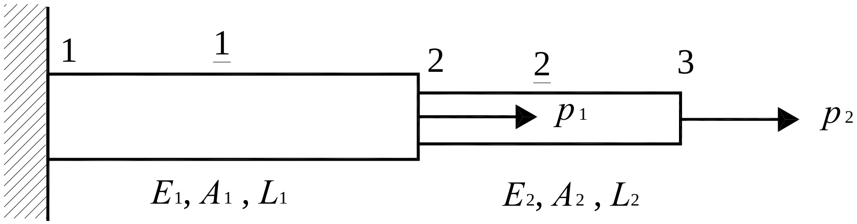
Overestimation in IFEM

- Multiple occurrences – element level
- Coupling – assemblage process
- Transformations – local to global and back
- Solvers – tightest enclosure
- Derived quantities – function of primary

Naïve interval FEA



$E_1 A_1 / L_1 = \boldsymbol{k}_1 = [0.95,\ 1.05],$
$E_2 A_2 / L_2 = \boldsymbol{k}_2 = [1.9,\ 2.1],$
$p_1 = 0.5, \quad p_2 = 1$

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow \begin{pmatrix} [2.85,\ 3.15] & [-2.1,\ -1.9] \\ [-2.1,\ -1.9] & [1.9,\ 2.1] \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

- exact solution: $\boldsymbol{u}_2$ = [1.429, 1.579], $\boldsymbol{u}_3$ = [1.905, 2.105]
- naïve solution: $\boldsymbol{u}_2$ = [−0.052, 3.052], $\boldsymbol{u}_3$ = [0.098, 3.902]
- interval arithmetic assumes that all coefficients are independent
- uncertainty in the response is severely overestimated (2000%)

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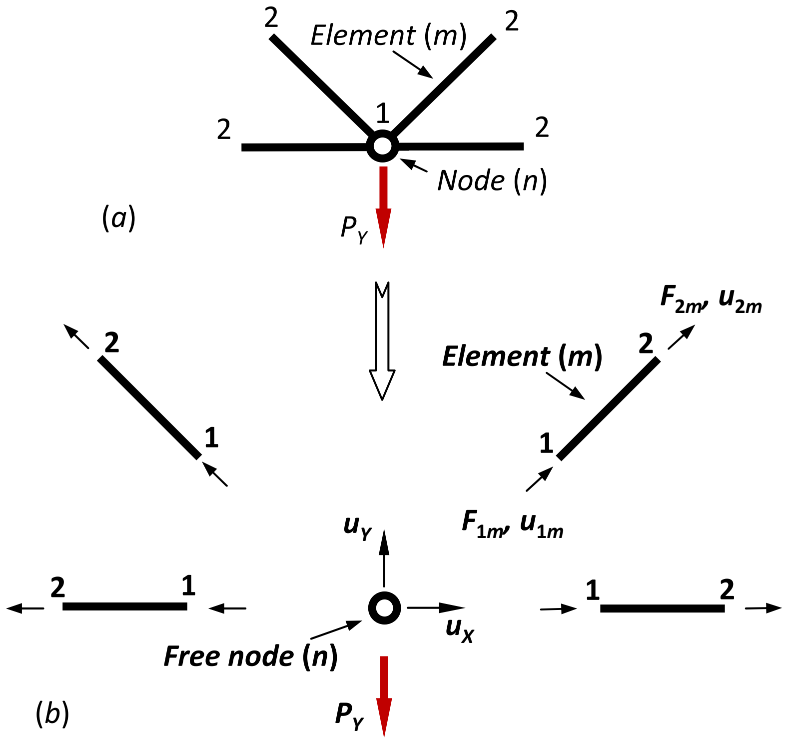
New Formulation



A typical node of a truss problem. (*a*) Conventional formulation. (*b*) Present formulation.

New Formulation

- Lagrange Multiplier Method

A method in which the minimum of a functional such as

$$I(u,v) = \int_a^b F(x,u,u',v,v')dx$$

with the linear equality constraints

$$G(u,u',v,v') = 0$$

is determined

New Formulation

- Lagrange Multiplier Method

The Lagrange's method can be viewed as one of determining *u*, *v* and λ by setting the first variation of the *modified* functional

$$L(u,v,\lambda) \equiv I(u,v) + \int_a^b \lambda G(u,u',v,v')dx = \int_a^b (F + \lambda G)dx$$

to zero

New Formulation

- Lagrange Multiplier Method

The result is Euler Equations of the $L(u,v,\lambda) \equiv \int_a^b (F+\lambda G)dx$

$$\left.\begin{aligned}
&\frac{\partial}{\partial u}(F+\lambda G)-\frac{d}{dx}\left[\frac{\partial}{\partial u'}(F+\lambda G)\right]=0\\
&\frac{\partial}{\partial v}(F+\lambda G)-\frac{d}{dx}\left[\frac{\partial}{\partial v'}(F+\lambda G)\right]=0\\
&G(u,u',v,v')=0
\end{aligned}\right\}$$

from which the dependent variables *u, v,* and λ can be determined at the same time

New Formulation

In steady-state analysis, the variational formulation for a discrete structural model within the context of Finite Element Method (FEM) is given in the following form of the total potential energy functional when subjected to the constraints $CU = V$

$$\Pi^* = \frac{1}{2} U^T K U - U^T P + \lambda^T (CU - V)$$

New Formulation

Invoking the stationarity of Π^*, that is $\delta\Pi^* = 0$, we obtain

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ V \end{pmatrix}$$

In order to force unknowns associated with coincident nodes to have identical values, the constraint equation $CU=V$ takes the form $CU = 0$, and the above system will have the following form

New Formulation

$$\begin{pmatrix} \mathbf{k} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix}$$

or

$$\mathbf{KU} = \mathbf{P}$$

where

New Formulation

$$\mathbf{k} = \begin{pmatrix} \mathbf{k}_1 & -\mathbf{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{k}_1 & \mathbf{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}_n & -\mathbf{k}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{k}_n & \mathbf{k}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{1X} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1Y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{mX} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{mY} \end{pmatrix}$$

$$\mathbf{k}_i = \frac{\mathbf{E}_i \mathbf{A}_i}{L_i}$$

New Formulation

$$\mathbf{u}_{1i} + \mathbf{u}_{jX} cos\varphi_i + \mathbf{u}_{jY} sin\varphi_i = 0$$

$$C^{\mathbf{T}} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ cos\,\varphi_1 & 0 & \cdots \\ sin\,\varphi_1 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & cos\,\varphi_1 & \cdots \\ 0 & sin\,\varphi_1 & \cdots \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} \mathbf{u}_{11} \\ \mathbf{u}_{21} \\ \vdots \\ \mathbf{u}_{1n} \\ \mathbf{u}_{2n} \\ \mathbf{u}_{1X} \\ \mathbf{u}_{1Y} \\ \vdots \\ \mathbf{u}_{mX} \\ \mathbf{u}_{mY} \end{pmatrix} \qquad \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_{11} \\ \boldsymbol{\lambda}_{21} \\ \vdots \\ \boldsymbol{\lambda}_{1n} \\ \boldsymbol{\lambda}_{2n} \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \mathbf{p}_{1X} \\ \mathbf{p}_{1Y} \\ \vdots \\ \mathbf{p}_{mX} \\ \mathbf{p}_{mY} \end{pmatrix}$$

New Formulation

- Iterative Enclosure (Neumaier 2007)

$$(K + B\,\mathbf{D}\,A)\mathbf{u} = a + F\,\mathbf{b}$$

$$\mathbf{v} = \{(ACa) + (ACF)\mathbf{b} + (ACB)\mathbf{d}\} \cap \mathbf{v}, \quad \mathbf{d} = \{(D_0 - \mathbf{D})\mathbf{v} \cap \mathbf{d}$$

$$\mathbf{u} = (Ca) + (CF)\mathbf{b} + (CB)\mathbf{d}$$

where

$$C := (K + BD_0 A)^{-1}$$

$$\mathbf{u} = Ca + CF\mathbf{b} + CB\mathbf{d}$$

$$\mathbf{v} = ACa + ACF\mathbf{b} + ACB\mathbf{d}$$

$$\mathbf{d} = (D_0 - \mathbf{D})\mathbf{v}$$

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Numerical example

- $$Width\ error\% = \left(\frac{computed\ enclosure\ width}{exact\ enclosure\ width} - 1 \right) \times 100$$
- $$Bound\ error\% = \left(\frac{computed\ bound - exact\ bound}{exact\ bound} \right) \times 100$$

Numerical example

- Eleven bar truss

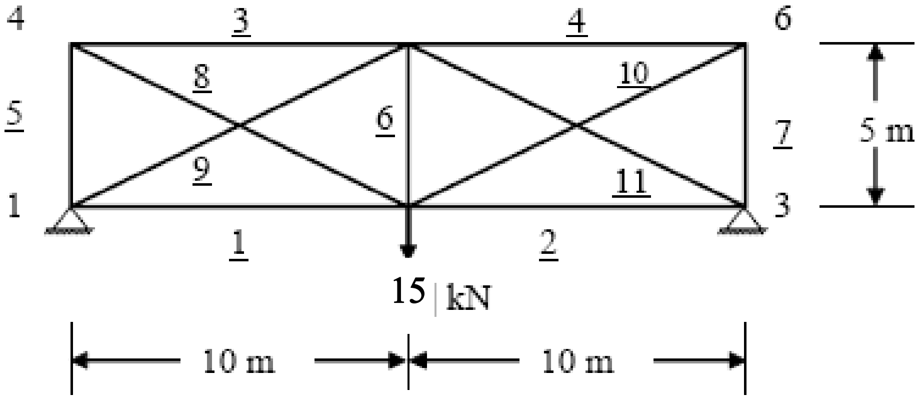
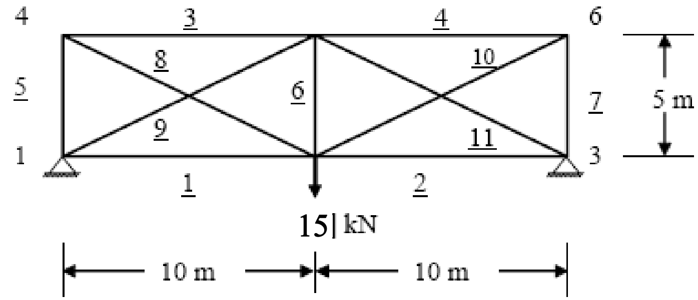


Table 2 Eleven bar truss -displacements for 12% uncertainty in the modulus of elasticity (E)						
	$V_2 \times 10^{-5}$		$U_4 \times 10^{-5}$		$V_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-15.903532	-14.103133	2.490376	3.451843	-0.843182	-0.650879
Krawczyk FPI	---	---	---	---	---	---
Neumaier's approach	-15.930764	-13.967877	2.431895	3.4943960	-0.848475	-0.633096
Error %(width)	9.02		10.50		11.99	
Present approach	-15.930764	-13.967877	2.431895	3.494396	-0.848475	-0.633096
Error %(width)	9.02		10.50		11.99	

Error in bounds%= 0.17 %



Numerical example

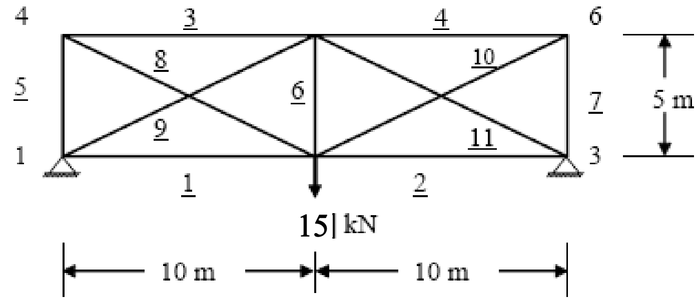
- Eleven bar truss

Table 4 Eleven bar truss - comparison of axial forces for 10% uncertainty in the modulus of elasticity (E) for various approaches

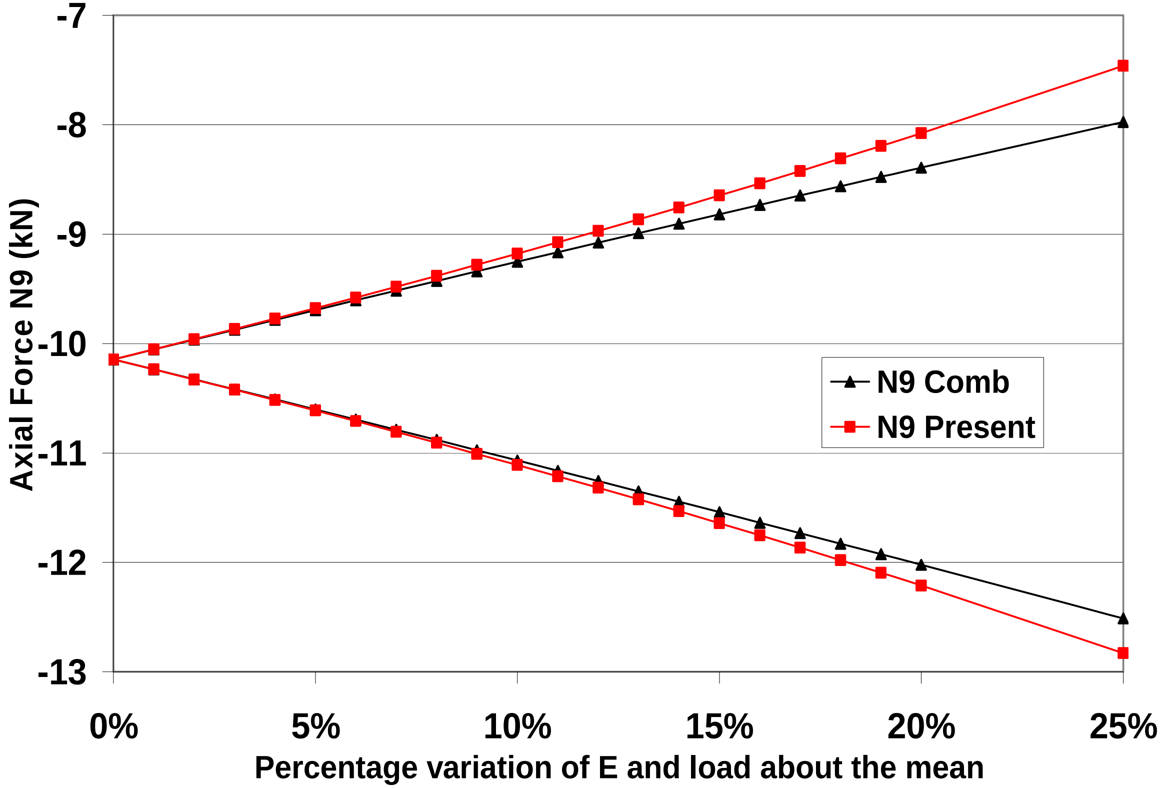
	$\underline{N}_3(kN)$	$\overline{N}_3(kN)$	$\underline{N}_9(kN)$	$\overline{N}_9(kN)$
Combinatorial approach	-6.28858	-5.57152	-10.54135	-9.73966
Simple enclosure $\mathbf{z}_1(\mathbf{u})$	-7.89043	-3.96214	-11.89702	-8.39240
Error %(width)	447.83		337.15	
Intersection $\mathbf{z}_2(\mathbf{u})$	-6.82238	-5.08732	-11.32576	-9.02784
Error %(width)	141.97		186.63	
Present approach	-6.31656	-5.53601	-10.58105	-9.70837
Error %(width)	8.85		8.85	

Error in bounds%= 0.45 %

Numerical example

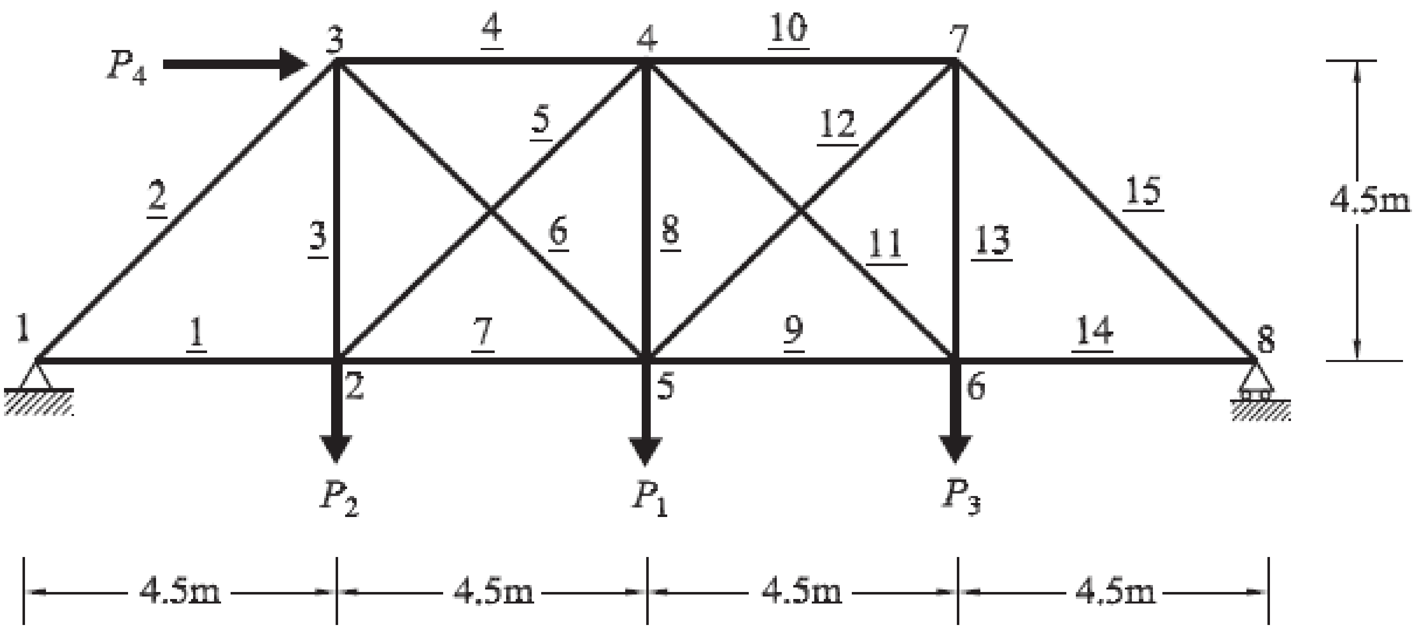


- Eleven bar truss – Bounds on axial forces

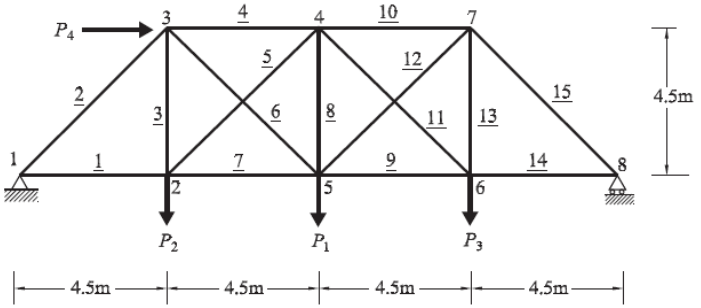


Numerical example

- Fifteen bar truss – Bounds on axial forces



Numerical example



- Fifteen bar truss – Bounds on axial forces

Table 12 Forces (kN) in elements of fifteen element truss for 10% uncertainty in modulus of elasticity (E) and load

Element	Combinatorial approach		Neumaier's approach		%Error in width	Present approach		%Error in width
	LB	UB	LB	UB		LB	UB	
1	254.125	280.875	227.375	310.440	210.53	254.125	280.875	0.000
2	-266.756	-235.289	-294.835	-210.187	169.01	-266.756	-235.289	0.000
3	108.385	134.257	95.920	148.174	101.97	107.098	134.987	7.797
4	-346.267	-302.194	-379.167	-272.461	142.12	-347.003	-300.909	4.585
5	-43.854	-16.275	-48.143	-12.985	27.48	-44.975	-14.543	10.344
14	211.375	233.625	189.125	258.217	210.53	211.375	233.625	0.000
15	-330.395	-298.929	-365.174	-267.463	210.53	-330.395	-298.929	0.000

Conclusions

- Development and implementation of IFEM
  - uncertain material, geometry and load parameters are described by interval variables
  - interval arithmetic is used to guarantee an enclosure of response
- Derived quantities obtained at the same accuracy of the primary ones
- The method is generally applicable to linear static FEM, regardless of element type
- IFEM forms a basis for generalized models of uncertainty in engineering