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RECURSIVE LEAST-SQUARES ESTIMATION IN CASE OF INTERVAL OBSERVATION DATA

Agenda

- Introduction
- Recursive least-squares estimation
- Observation imprecision
- Interval extension of recursive estimation
- Application example
- Conclusions

Introduction

- Monitoring of processes
  - Production / construction
  - Geodynamics
  - …
- Observation of process features
  - Quantitative ⇔ Qualitative
- Analysis tasks
  - Modeling
  - Prediction and filtering
  - Parameter identification

} Real-time requirements

Introduction

- Models
  - Causality ⇔ Correlation
  - Structure ⇔ Behaviour
  - Statics ⇔ Dynamics
- Parameters
  - System state ⇔ additional parameters
  - Time variant ⇔ time invariant
- Observations
  - Direct ⇔ indirect

Highly flexible application
⇩
Example

Starting point

Linear / linearized model (Gauss-Markov)

Functional model

Observation equations: mean values ⇨ geometry & physics

$$E(\underline{\mathbf{l}}) = \mathbf{A}\,\mathbf{x} \quad \left(\text{or } E(\underline{\Delta\mathbf{l}}) = \mathbf{A}\,\Delta\mathbf{x},\right.$$

$$\left.\text{with } \underline{\Delta\mathbf{l}} := \underline{\mathbf{l}} - \mathbf{f}(\mathbf{x}_0),\ \Delta\mathbf{x} := \mathbf{x} - \mathbf{x}_0,\ \mathbf{A} := \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\mathbf{x}_0}\right)$$

Residual equations

$$\mathbf{v} = \mathbf{A}\,\mathbf{x} - \mathbf{l} \quad \text{with } \mathbf{v} = E(\underline{\mathbf{l}}) - \mathbf{l}$$

Stochastic model

Dispersion matrix: random variability ⇨ uncertainty

$$V(\underline{\mathbf{l}}) = \mathbf{\Sigma}_{ll} = \mathbf{\Sigma}_{vv} = \sigma_0^2\mathbf{Q}_{ll} = \sigma_0^2\mathbf{P}^{-1}$$

Recursive least-squares estimation

Classical weighted least-squares solution

LS-principle

$$\mathbf{v}^T\mathbf{P}\,\mathbf{v} \rightarrow \min \quad \Rightarrow \quad \hat{\mathbf{x}}, \hat{\mathbf{l}}, \mathbf{v}, \hat{\sigma}_0^2$$

Normal equations

$$\mathbf{A}^T\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{P}\mathbf{l}$$

- Regular system
- Singular system

Estimated parameters

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{P}\mathbf{l}$$

Dispersion matrix

$$\mathbf{Q}_{\hat{x}\hat{x}} = \left(\mathbf{A}^T\mathbf{P}\mathbf{A}\right)^{-1} \text{ and } \Sigma_{\hat{x}\hat{x}} = \sigma_0^2\mathbf{Q}_{\hat{x}\hat{x}}$$

Recursive least-squares estimation

Epoch / step index

Recursive formulation $\rightarrow \hat{\mathbf{x}}^{(i-1)}, \mathbf{l}^{(i)}$

Uncorrelated epochs

Separation of l

$$\begin{bmatrix} \mathbf{v}^{(i-1)} \\ \mathbf{v}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(i-1)} \\ \mathbf{A}^{(i)} \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{l}^{(i-1)} \\ \mathbf{l}^{(i)} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{P}^{(i-1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{(i)} \end{bmatrix}$$

Estimated parameters

$$\hat{\mathbf{x}}^{(i)} = \left(\left(\mathbf{A}^{(i-1)}\right)^T \mathbf{P}^{(i-1)} \mathbf{A}^{(i-1)} + \left(\mathbf{A}^{(i)}\right)^T \mathbf{P}^{(i)} \mathbf{A}^{(i)} \right)^{-1} \ldots$$

$$\ldots \left(\left(\mathbf{A}^{(i-1)}\right)^T \mathbf{P}^{(i-1)} \mathbf{l}^{(i-1)} + \left(\mathbf{A}^{(i)}\right)^T \mathbf{P}^{(i)} \mathbf{l}^{(i)} \right)$$

Dispersion matrix

$$\mathbf{Q}_{\hat{x}\hat{x}}^{(i)} = \left(\left(\mathbf{A}^{(i-1)}\right)^T \mathbf{P}^{(i-1)} \mathbf{A}^{(i-1)} + \left(\mathbf{A}^{(i)}\right)^T \mathbf{P}^{(i)} \mathbf{A}^{(i)} \right)^{-1}$$

Recursive least-squares estimation

Recursive formulation – continued

Matrix identity

$$\left(\mathbf{A}^{-1}+\mathbf{B}\,\mathbf{D}^{-1}\mathbf{C}\right)^{-1}=\mathbf{A}-\mathbf{A}\mathbf{B}\left(\mathbf{D}+\mathbf{C}\,\mathbf{A}\,\mathbf{B}\right)^{-1}\mathbf{C}\,\mathbf{A}$$

Estimated parameters

$$\hat{\mathbf{x}}^{(i)}=\mathbf{x}^{(i-1)}+\mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)}\left(\mathbf{A}^{(i)}\right)^{T}\left(\mathbf{Q}_{ww}^{(i)}\right)^{-1}\mathbf{w}^{(i)}$$

Auxiliary quantities

$$\mathbf{w}^{(i)}=\mathbf{l}^{(i)}-\mathbf{A}^{(i)}\hat{\mathbf{x}}^{(i-1)},$$

Inconsistency vector

$$\mathbf{Q}_{ww}^{(i)}=\mathbf{Q}_{ll}^{(i)}+\mathbf{A}^{(i)}\mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)}\left(\mathbf{A}^{(i)}\right)^{T}$$

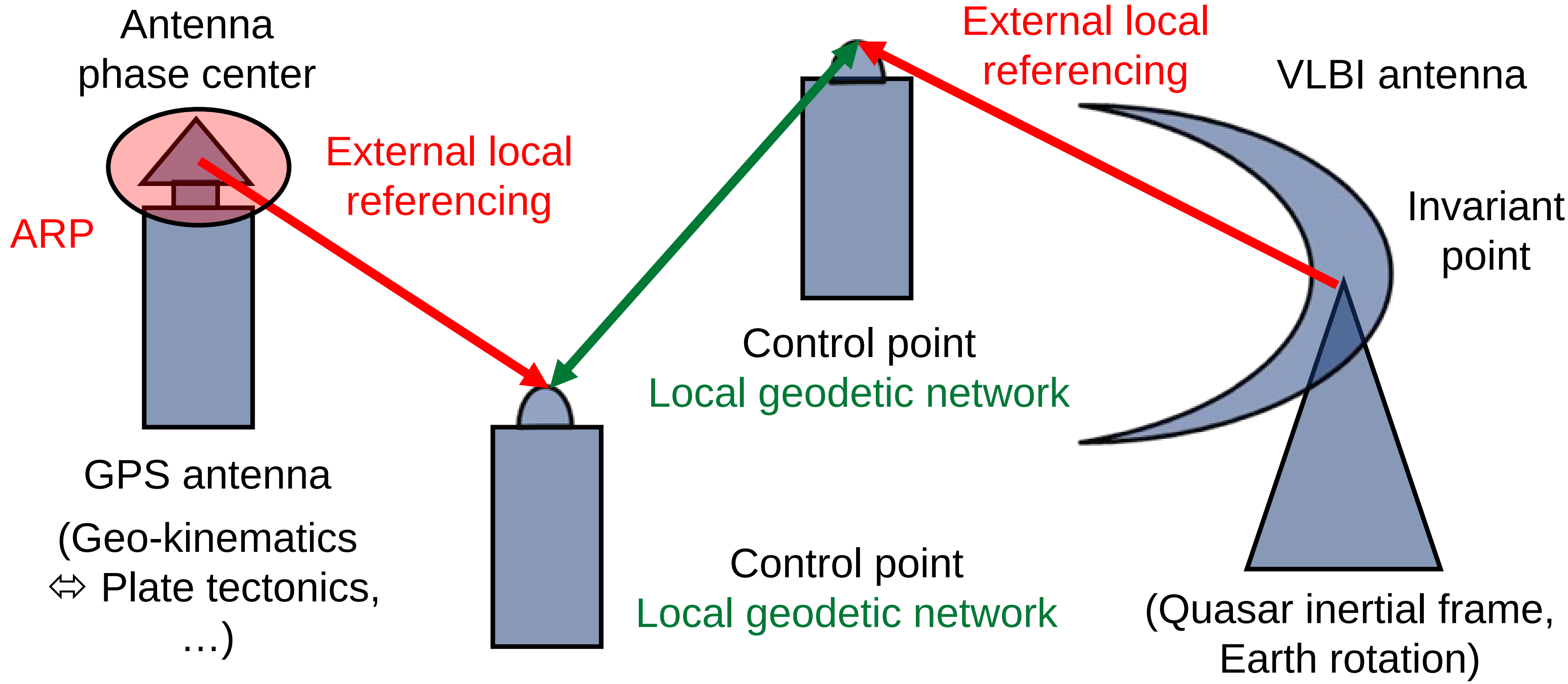
Dispersion matrix

$$\mathbf{Q}_{\hat{x}\hat{x}}^{(i)}=\mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)}-\mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)}\left(\mathbf{A}^{(i)}\right)^{T}\left(\mathbf{Q}_{ww}^{(i)}\right)^{-1}\mathbf{A}^{(i)}\mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)}$$

Observation imprecision

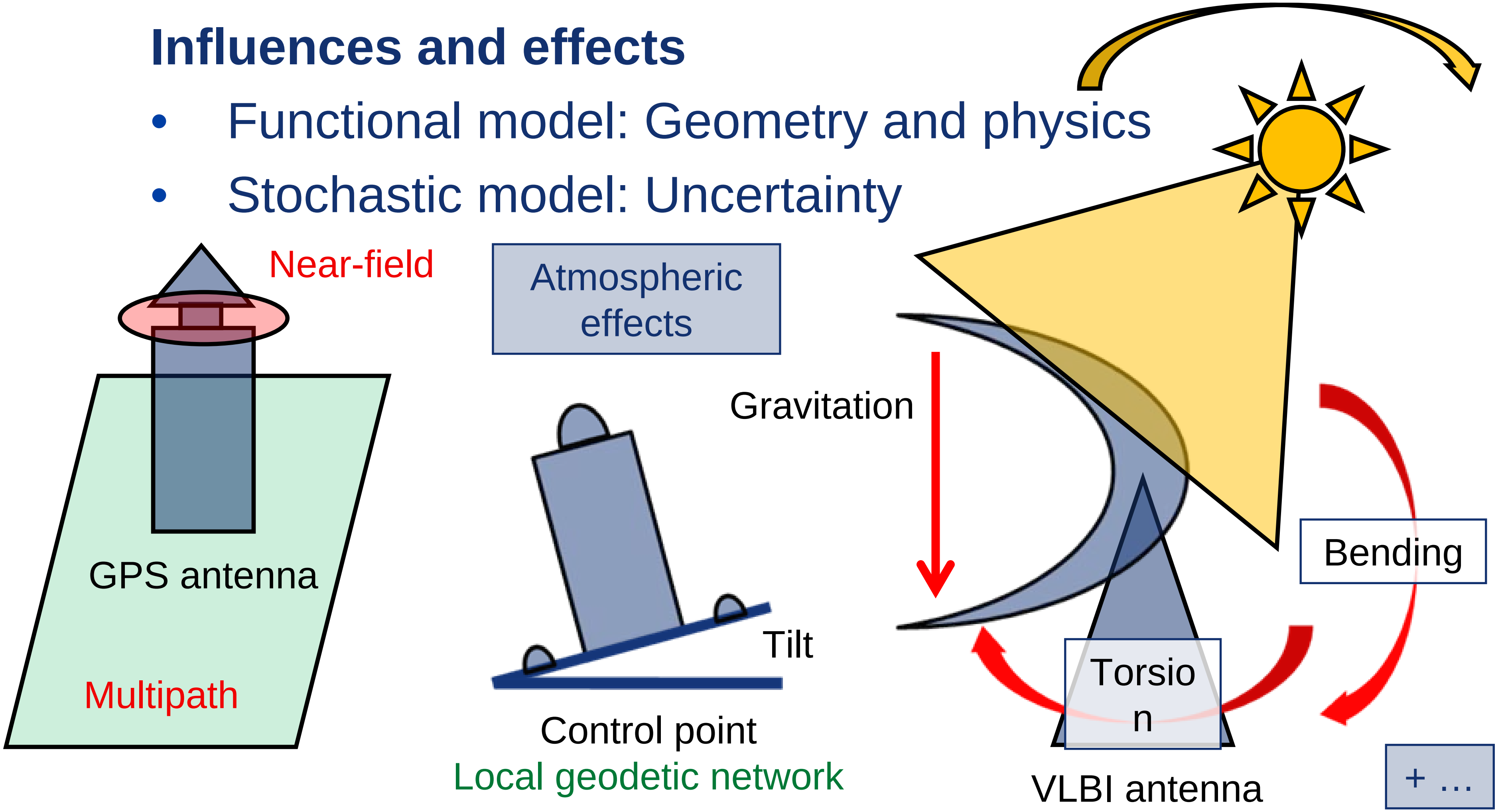
Geodetic observations

Example: Global terrestrial reference frames



Influences and effects

- Functional model: Geometry and physics
- Stochastic model: Uncertainty

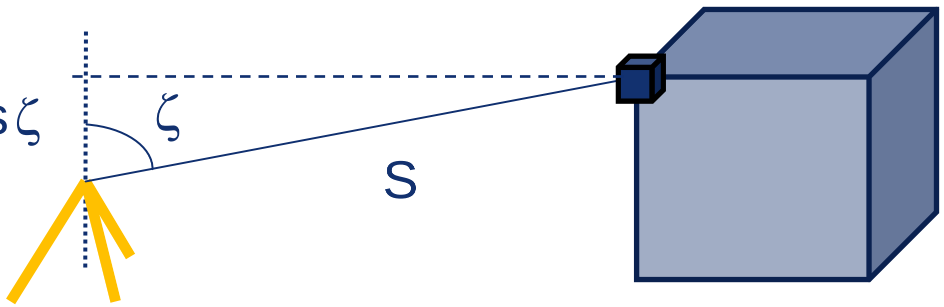


Specific modeling of observations

Example: Zenith angles

Extended error model for total stations

$$\Delta h = S\cos\zeta$$



$$\zeta' = \underbrace{\left(\zeta + i_{Ex} + c_v + L(r)\right)}_{\text{Internal corrections}} - \underbrace{\frac{1}{2}\frac{\sin\zeta\, S}{n_A}\frac{\partial n_A}{\partial z'}\rho}_{\text{Correction of refraction}} - \underbrace{\frac{\sin\alpha_A}{S}\left(\frac{n_A\cos\alpha_A d}{\sqrt{n_R^2 - n_A^2\sin^2\alpha_A}} - e\right)\rho}_{\text{Impact of angle of incidence}}$$

Observation imprecision

Modeling and pre-processing techniqes

- Mitigation / prevention of the influences and effects, respectively
- Testing and calibration
- Correction models
- Linear combination of observations
- Parametrization and estimation
- …

Remaining effects: random + systematic

Observation imprecision

Modeling of influences

⇨ Re-interpretation of the observation vector

Original observation

Functional model

Influence parameters

Correction model

$$\mathbf{l} = \bar{\mathbf{y}} + \mathbf{g}(\mathbf{s})$$

Lineari-zation

$$\approx \bar{\mathbf{y}} + \mathbf{g}(\mathbf{s}_0) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right|_{\mathbf{s}_0} (\mathbf{s} - \mathbf{s}_0)$$

$$= \mathbf{y} + \mathbf{G}\,\Delta\mathbf{s}$$

Approximate values

Corrected observations

$$\text{with } \mathbf{y} := \bar{\mathbf{y}} + \mathbf{g}(\mathbf{s}_0), \mathbf{G} := \left.\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right|_{\mathbf{s}_0}$$

Modeling of influences: Examples

Individual influences

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \Delta s_1 \\ \Delta s_2 \\ \vdots \\ \Delta s_n \end{pmatrix}$$

Offset

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \Delta s$$

Drift

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} t_1 - t_0 \\ t_2 - t_0 \\ \vdots \\ t_n - t_0 \end{pmatrix} \Delta s$$

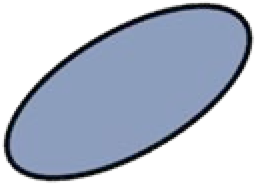
Scale

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \Delta s$$

Observation uncertainty: Concepts

Random variable

$$\underline{\mathbf{y}} \sim N\left(\mu_y, \Sigma_{yy}\right)$$



Random variability

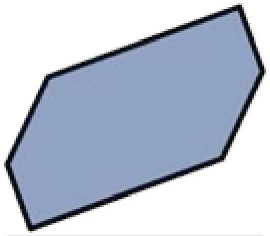
Imprecision (Intervals)

Deterministic variable

$$[\Delta\mathbf{s}] = [-\mathbf{s}_r, +\mathbf{s}_r] = \langle \mathbf{0}, \mathbf{s}_r \rangle$$

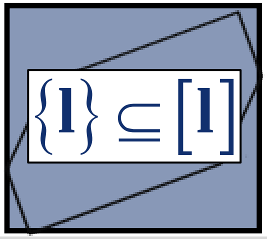
True range of values

$$\{\mathbf{l}\} = \{\mathbf{l} = \mathbf{y} + \mathbf{G}\,\Delta\mathbf{s} \,|\, \Delta\mathbf{s} \in [\Delta\mathbf{s}]\}$$



Interval inclusion

$$[\mathbf{l}] = \langle \mathbf{l}_m, \mathbf{l}_r \rangle = \langle \mathbf{y}, |\mathbf{G}|\mathbf{s}_r \rangle$$

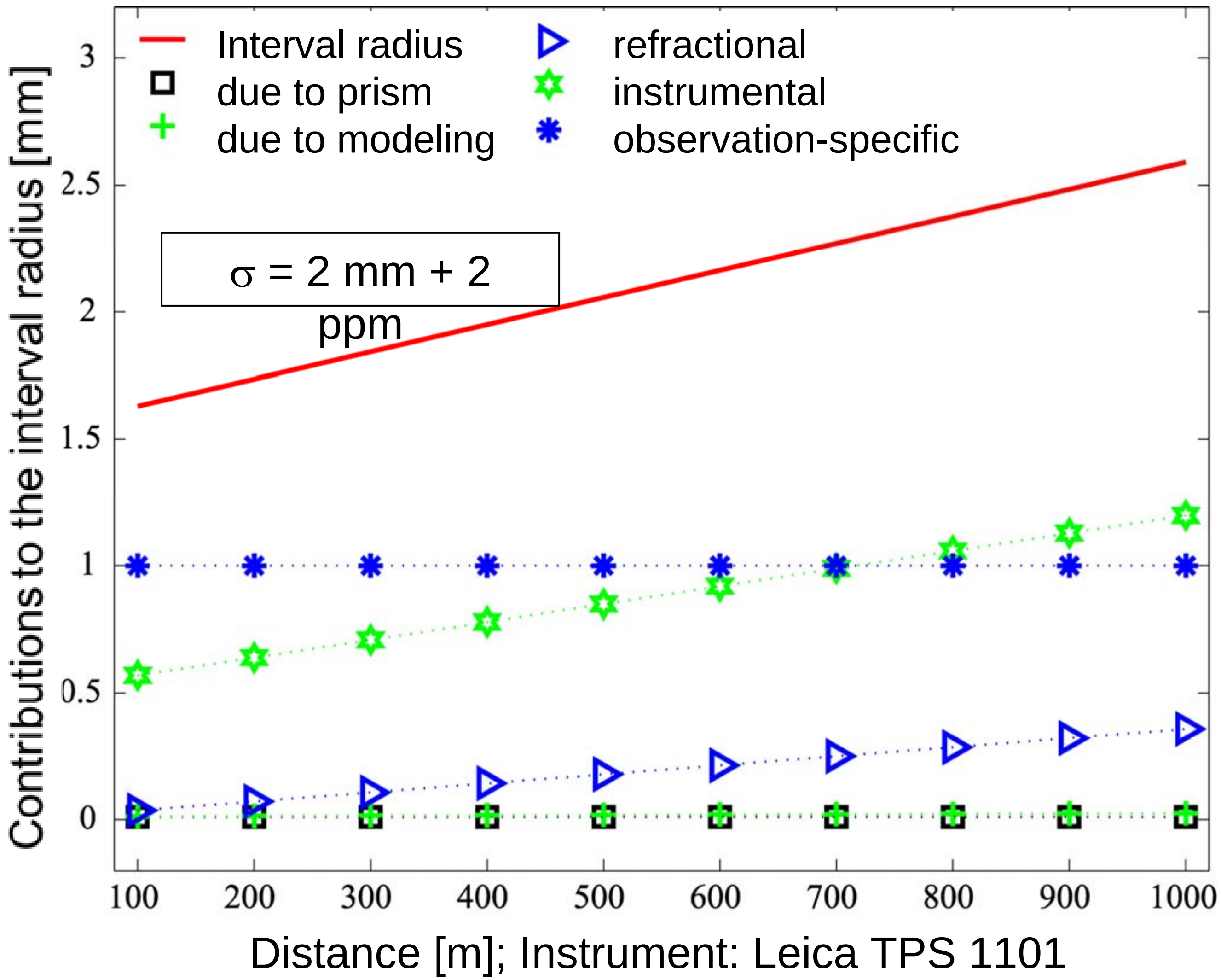


Observation imprecision

Example:

Imprecision model for distance observations

Distance-dependent interval radii



Interval evaluation of the real-valued equations

Interval bounds too pessimistic

$$\left[\hat{\mathbf{x}}^{(i)}\right] := \left[\mathbf{x}^{(i-1)}\right] + \left(\mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)}\left(\mathbf{A}^{(i)}\right)^{T}\left(\mathbf{Q}_{ww}^{(i)}\right)^{-1}\right)\left[\mathbf{w}^{(i)}\right],$$

$$\left[\mathbf{w}^{(i)}\right] = \left[\mathbf{l}^{(i)}\right] - \mathbf{A}^{(i)}\left[\hat{\mathbf{x}}^{(i-1)}\right]$$

Exemplary characterisation and treatment

$$[\mathrm{x}]-[\mathrm{x}]=?$$

$$[\mathrm{x}]-[\mathrm{x}]=[-2\mathrm{x}_{\mathrm{r}},+2\mathrm{x}_{\mathrm{r}}]$$ Naïve approach

$$\left((1,-1)\begin{pmatrix}1\\1\end{pmatrix}\right)[\mathrm{x}]=0[\mathrm{x}]=0$$ True range of values

Interval extension of recursive estimation

Problem of overestimation

Interval inclusion

$$\{\mathbf{z}\} = \left\{\mathbf{z} \middle| \mathbf{z} = \mathbf{F}\,\mathbf{x}, \mathbf{x} \in [\mathbf{x}]\right\} \subseteq [\mathbf{z}]_0 = \mathbf{F}[\mathbf{x}]$$

Subdistri-butivity

$$\mathbf{z} = \mathbf{F}\,\mathbf{x}, \quad \mathbf{x} \in [\mathbf{x}] \quad \Rightarrow \quad (\mathbf{M}\,\mathbf{F})[\mathbf{x}] \subseteq \mathbf{M}\left(\mathbf{F}[\mathbf{x}]\right)$$

Correct range of values ⇔ Interval inclusion

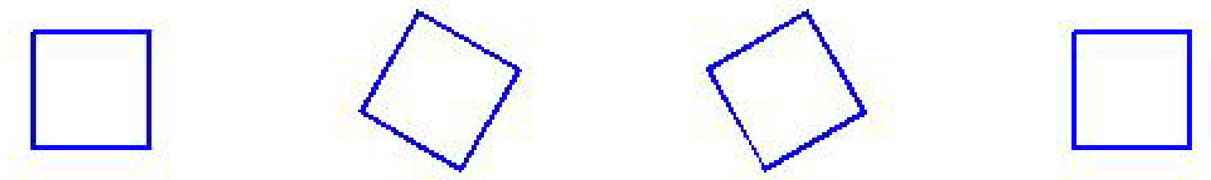
$$\left\{\hat{\mathbf{x}}^{(i)}\right\} = \left\{\mathbf{x}^{(i)} \middle| \hat{\mathbf{x}}^{(i-1)} + \mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)} \left(\mathbf{A}^{(i)}\right)^T \left(\mathbf{Q}_{ww}^{(i)}\right)^{-1} \left(\mathbf{l}^{(i)} - \mathbf{A}^{(i)}\mathbf{x}^{(i-1)}\right),\right.$$

$$\left.\ldots\, \mathbf{l}^{(i)} \in \left[\mathbf{l}^{(i)}\right], \hat{\mathbf{x}}^{(i-1)} \in \left\{\mathbf{x}^{(i-1)}\right\}\right\} \subseteq \left[\hat{\mathbf{x}}^{(i)}\right]$$

Interval extension of recursive estimation

Rotation of an interval vector (unit square) in 30° steps

Pure rotation



$\alpha = 0°$ $\alpha = 30°$ $\alpha = 60°$ $\alpha = 90°$

Interval inclusion of the true range of values

⇨ First: Interval midpoints

$$\hat{\mathbf{x}}_m^{(i)} = \mathbf{x}_m^{(i-1)} + \mathbf{Q}_{\hat{x}\hat{x}}^{(i-1)} \left(\mathbf{A}^{(i)}\right)^T \left(\mathbf{Q}_{ww}^{(i)}\right)^{-1} \mathbf{w}_m^{(i)}$$

$$\mathbf{w}_m^{(i)} = \mathbf{l}_m^{(i)} - \mathbf{A}^{(i)} \hat{\mathbf{x}}_m^{(i-1)}$$

Interval extension of recursive estimation

⇨ Second: Efficient calculation of the tightest radii

Starting point

$$\hat{\mathbf{x}}^{(i)} = \mathbf{Q}_{\hat{x}\hat{x}}^{(i)} \left(\left(\mathbf{A}^{(i-1)} \right)^T \mathbf{P}^{(i-1)} \mathbf{l}^{(i-1)} + \left(\mathbf{A}^{(i)} \right)^T \mathbf{P}^{(i)} \mathbf{l}^{(i)} \right)$$

Reference to influence parameters

$$\hat{\mathbf{x}}^{(i)} = \mathbf{Q}_{\hat{x}\hat{x}}^{(i)} \left(\left(\mathbf{A}^{(i-1)} \right)^T \mathbf{P}^{(i-1)} \left(\mathbf{y}^{(i-1)} + \mathbf{G}^{(i-1)} \Delta \mathbf{s} \right) + \ldots \right.$$

$$\left. \left(\mathbf{A}^{(i)} \right)^T \mathbf{P}^{(i)} \left(\mathbf{y}^{(i)} + \mathbf{G}^{(i)} \Delta \mathbf{s} \right) \right)$$

„Global“ set of original influence parameters

Refined observation model as introduced

Efficient calculation of the tightest radii – continued

Reformulation

$$\hat{\mathbf{x}}^{(i)} = \mathbf{x}_m^{(i)} + \mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}^{(i)} \left(\left(\mathbf{A}^{(i-1)}\right)^T \mathbf{P}^{(i-1)} \mathbf{G}^{(i-1)} + \left(\mathbf{A}^{(i)}\right)^T \mathbf{P}^{(i)} \mathbf{G}^{(i)} \right) \Delta \mathbf{s}$$

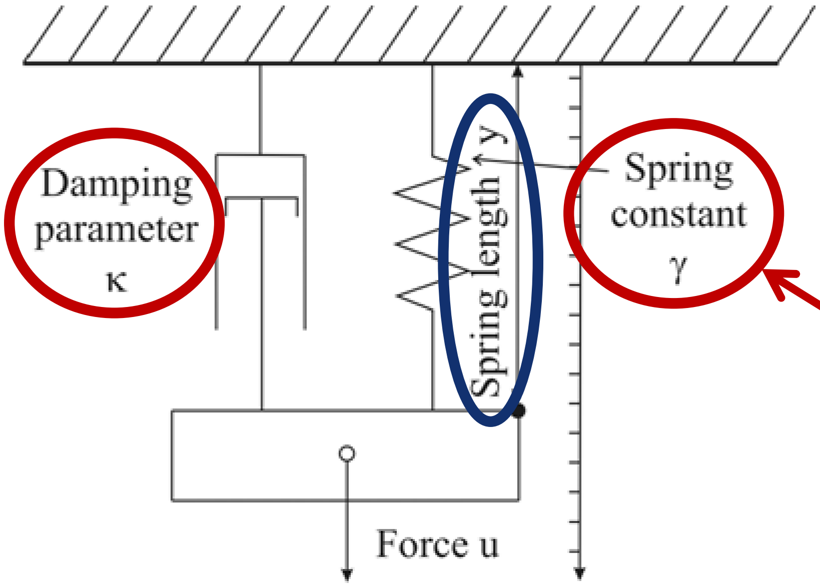
Interval radius

$$\hat{\mathbf{x}}_r^{(i)} = \left| \mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}^{(i)} \underbrace{\left(\left(\mathbf{A}^{(i-1)}\right)^T \mathbf{P}^{(i-1)} \mathbf{G}^{(i-1)} + \left(\mathbf{A}^{(i)}\right)^T \mathbf{P}^{(i)} \mathbf{G}^{(i)} \right)}_{\mathbf{M}^{(i)}} \right| \mathbf{s}_r$$

Efficient, recursive calculation

Overestimation avoided

Damped harmonic oscillation



$$y(t) = y_0 + A\exp(-\kappa t)\sin\left(\frac{2\pi}{T}t + \varphi\right)$$

A	Amplitude
φ	Phase
κ	Damping parameter
T	Period length
y_0	Offset

Numerical simulation: Formulation as a recursive estimation

Damped harmonic oscillation

Modeling alternatives

Measurement standard deviations

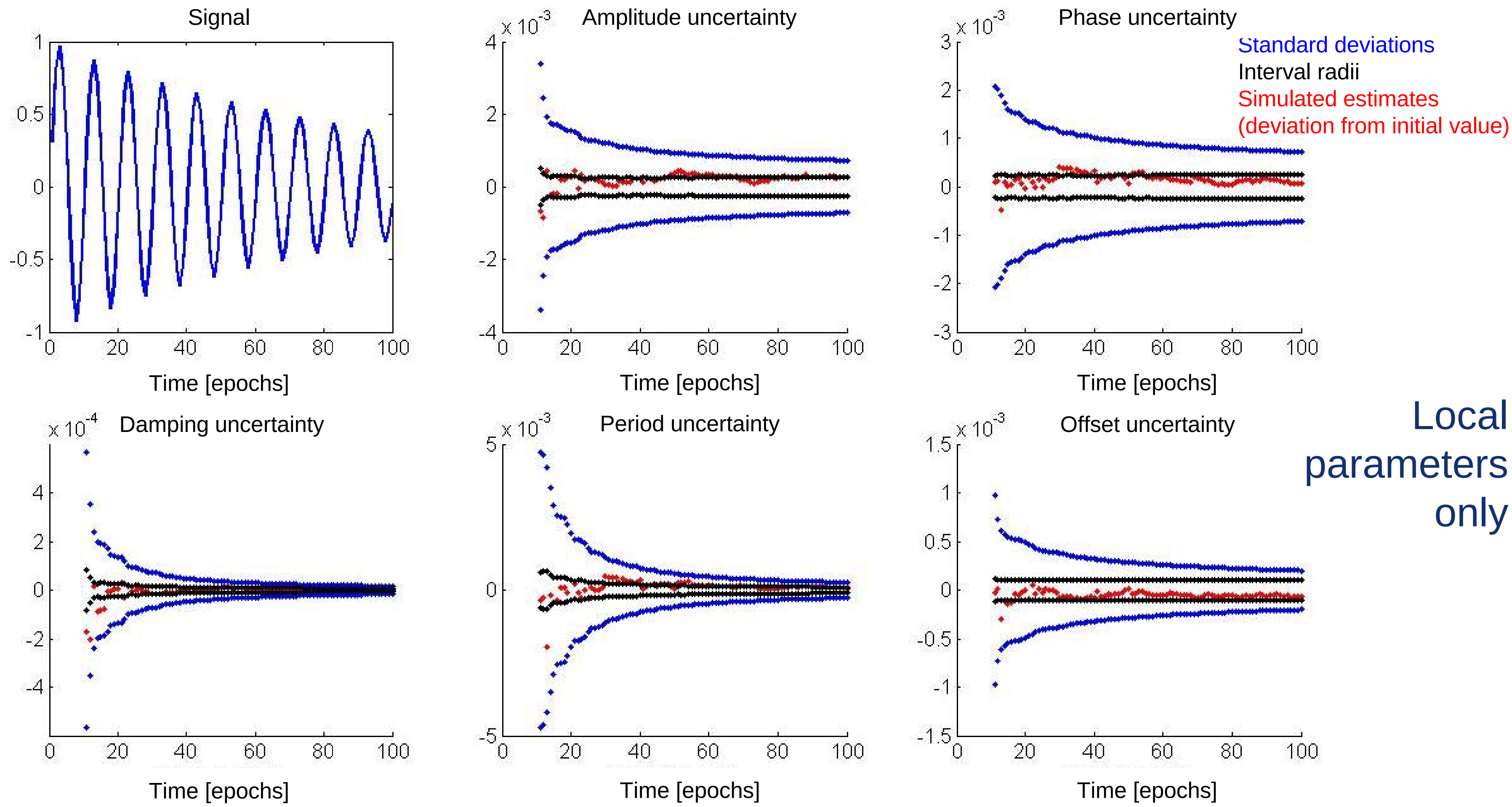
Simulation parameters

A	ϕ	κ	T	y_0	σ
1	$-\pi/10$	0.01	10	0	0.001

Imprecision modeling (interval radii)

- Alternative I: local parameters for each observation ($s_r=10^{-4}$)
- Alternative II: global additive parameter ($s_r=10^{-3}$), global parameter proportional with y ($s_r=10^{-4}$), no local parameters
- Alternative III: as Alternative II + global parameter prop. with t ($s_r=10^{-4}$)

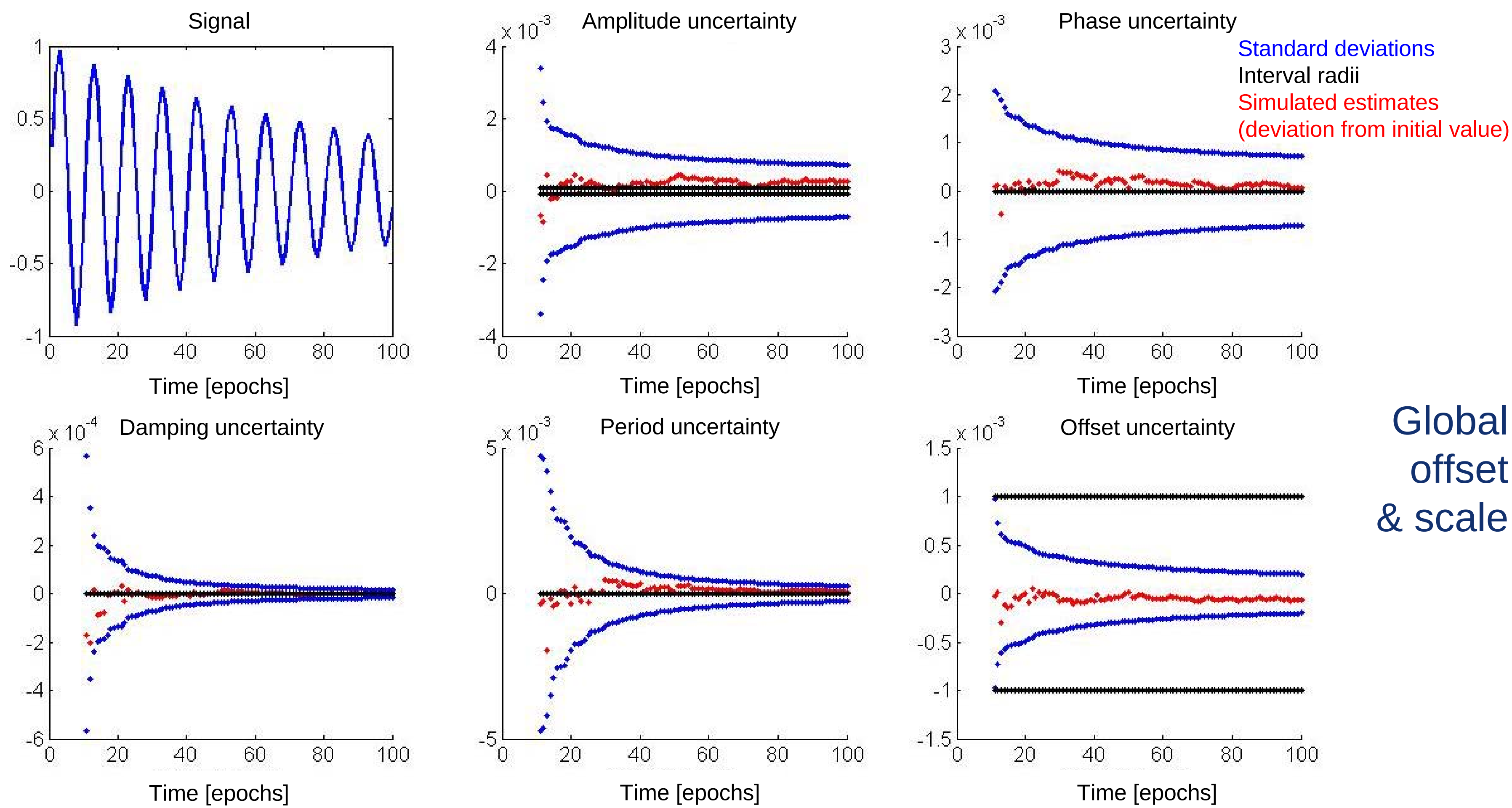
Damped harmonic oscillation – recursive estimation (Alternative I)



Local parameters only

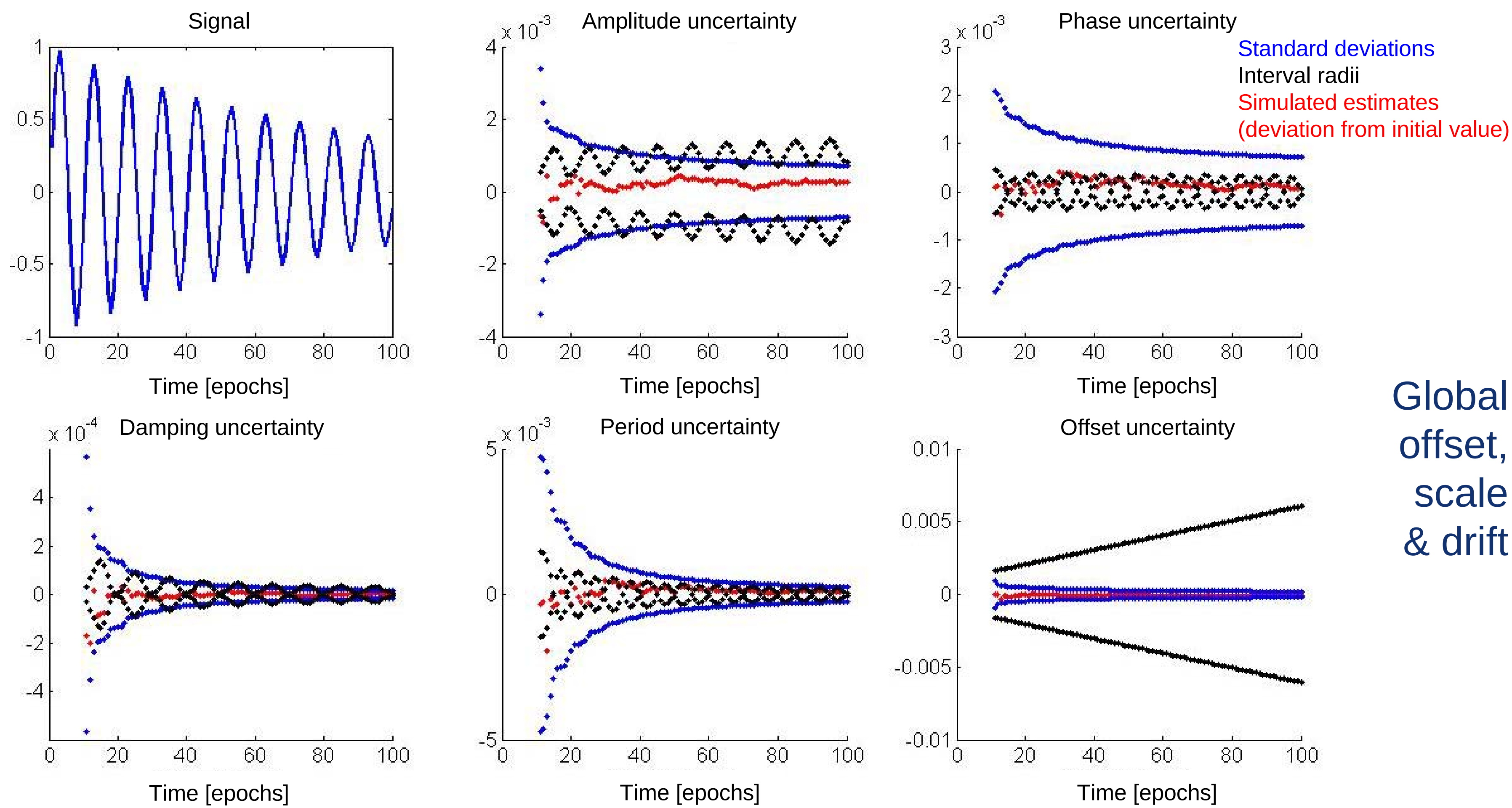
Application example

Damped harmonic oscillation – recursive estimation (Alternative II)



Application example

Damped harmonic oscillation – recursive estimation (Alternative III)



Global offset, scale & drift

Conclusions

- Recursive estimation is relevant in real-time applications
- Consideration of imprecision is possible
  - Reference to influence parameters
  - Overestimation problem
- Propagation of interval midpoints as in the real case
- Propagation of interval radii
  - either by resolving the recursion (inefficient!)
  - or by referring to recursively updated matrices (efficient!)
- Given example: Different characteristics of dispersion and imprecision (⇨ use for design and diagnostics)
- Outlook: Application to state-space filtering

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