Abstract: A real tunnel project was used to demonstrate the validity of RS-FEM framework in geomechanical problems where engineers have to face not only imprecise data but also lack of information at the beginning of the project. The objective of this paper is to illustrate the RS-FEM approach, which in the authors’ view is an appealing procedure in uncertainty analysis, by means of a real case history validated by in-situ measurements. Furthermore, some advantages and limitations in geotechnical problems are addressed as well as elaborating its procedure, which includes providing set-based input random variables, computing probability assignment of deterministic FE calculations and demonstrating the usage of such a framework in reliability analysis.

Keywords: Uncertainty, Tunnelling, Reliability Analysis, Random Set theory, Finite Element Method

1. Introduction

“Since the early 1990s, no other area of the construction industry has been as adversely affected by major losses as tunnelling. Besides property losses often in the two-digit million range, third-party liability losses have also been high, and numerous people have lost their lives. The international insurance industry has made payments exceeding US$ 600m for large losses” (Wannick 2007). Certainly a variety of causes may be included for the losses such as flood, earthquake or fire. However, causes originating from uncertainties, lack of knowledge or insufficient data prior to tunnel construction cannot be overlooked. Therefore, modern concepts dealing with uncertainties (e.g. reliability analysis, risk management and sensitivity analysis) have to be introduced into common engineering practice, especially in big underground structures. This requires an efficient user-friendly framework dealing with uncertainties. In this respect, Random Set Finite Element Method (RS-FEM) has been recognized by the authors as a possible framework.

The random set theory developed by several authors (e.g. Dempster 1967, Kendall 1974, Shafer 1976, Dubois 1991) has provided an appropriate mathematical model to cope with uncertainty overcoming some of the drawbacks of "classical" probability theory. Recently Tonon et al. (1996, 2000a,b) demonstrated the application of Random Set Theory (RST) in rock mechanics and reliability analysis of a tunnel lining. Following them Peschl (2004), Schweiger and Peschl (2007) have extended Random Set Theory to be combined with the finite element method, called Random-Set-Finite-Element-Method (RS-FEM). They illustrated the applicability of the developed framework to practical geotechnical problems and showed that RS-FEM is an efficient tool for reliability analysis in geotechnics in early design phases being highly
complementary to the so-called observational method. For further details about basic concepts of RST and RS-FEM procedure the reader is referred to the work of e.g. Tonon & Mammino 1996, Schweiger & Peschl 2007.

2. Project Description and Input Parameters

Here a tunnel application was chosen to demonstrate the efficiency and validity of the RS-FEM. This 460 m long tunnel located in Germany with the typical horse-shoe shaped section and dimension of $15 \times 12.3 \text{m}$ width and height respectively, is constructed according to the principles of the New Austrian Tunnelling Method (NATM), and is divided into three main excavation stages; top-heading, bench and invert. The overburden along the tunnel axis starts from 7.5m in the portal region to a maximum of 25m. A section with the overburden of 25 m was selected to apply the RS-FEM. The relevant 2D model geometry and finite element mesh including some model specifications are depicted in Figure (1). Approximately 900 15-noded triangular elements were employed in our model. The finite element software Plaxis® has been used for all calculations (Brinkgreve et al. 2008).

![Figure 1. Deterministic finite element mesh used in RS-FEM.](image)

It follows from the geotechnical report that the aforementioned tunnel section is located in a homogeneous weathered and loosened sedimentary rock mass with a range of characteristics values for a generalized Hoek-Brown (HB) criterion as listed in Table (1).
Table 1. Hoek-Brown model parameters of the rock mass derived from site investigation report.

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Symbol</th>
<th>Unit weight</th>
<th>Elastic modulus of intact rock</th>
<th>Rock mass rating</th>
<th>Geological strength index</th>
<th>Unconfined compressive strength of intact rock</th>
<th>HB parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>γ</td>
<td>kN/m³</td>
<td>Ei</td>
<td>RMR</td>
<td>GSI</td>
<td>σci</td>
<td>m_i</td>
</tr>
</tbody>
</table>

Table 2. Parameters for structural elements.

<table>
<thead>
<tr>
<th>Structural Element</th>
<th>Location</th>
<th>Behaviour</th>
<th>Type</th>
<th>EA (kN/m)</th>
<th>EI (kN.m²/m)</th>
<th>Plastic Limit Force</th>
<th>Plastic Limit Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shotcrete Top-heading</td>
<td>Elastic Beam</td>
<td>1.5E6</td>
<td>1.12E4</td>
<td>-</td>
<td></td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Shotcrete Bench</td>
<td>Elastic Beam</td>
<td>1.5E6</td>
<td>1.12E4</td>
<td>-</td>
<td></td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Shotcrete Invert</td>
<td>Elastic Beam</td>
<td>1.0E6</td>
<td>3333.3</td>
<td>-</td>
<td></td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Anchor Elasto-plastic</td>
<td>Geogrid</td>
<td>79.8E3</td>
<td>-</td>
<td>153</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Providing Set-Based Input For Random Set Analysis

3.1. RANDOM SET INPUT OF ELASTIC MODULUS:

There are many empirical equations and correlations between some important geological indices e.g. RMR and GSI in the literature (see e.g. Zhang 2005, Hoek 2006), describing the condition of the rock mass, and elastic modulus of intact rock. For instance Eq. (1) given by Hoek 1997 was chosen to estimate the elastic modulus of the rock mass.

\[ E_m = \sqrt{\frac{\sigma_{ci}}{100}}10^{GSI^{-10}/40} \quad \text{(GPa)} \quad \text{for rocks: } \sigma_{ci} \leq 100 \]  

(1)

Considering the ranges of GSI and \( \sigma_{ci} \) presented in Table (1), one is able to infer two sets of parameters for the geological condition of the rock mass assuming that the lower value of UCS combined with the range of GSI makes up the first set and similarly the upper value of UCS forms the second set. Consequently, in this manner two random sets given in Table (3) are derived for the elastic modulus of rock mass using Eq. (1).

Table 3. Input random sets of elastic modulus of rock mass.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Probability of assignment</th>
<th>( \sigma_{ci} ) (UCS)</th>
<th>GSI</th>
<th>( E_m ) (modulus of rock mass)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td>-</td>
<td>MPa</td>
<td>-</td>
<td>MPa</td>
</tr>
<tr>
<td>Set No. 1</td>
<td>0.5</td>
<td>10</td>
<td>30-40</td>
<td>1000-1770</td>
</tr>
<tr>
<td>Set No. 2</td>
<td>0.5</td>
<td>50</td>
<td>30-40</td>
<td>2230-3970</td>
</tr>
</tbody>
</table>
3.2. RANDOM SET INPUT OF STRENGTH PARAMETERS:

Since in the original design, the Mohr-Coulomb (MC) failure criterion has been employed, this is maintained in the RS-FEM analysis. The HB parameters were in hand from the geotechnical report. In contrast to MC, the HB failure criterion is non-linear in principal stress space. Hence, to obtain a similar response with both models, it is necessary to fit equivalent MC parameters the part of the HB failure line which is close to the stress range of the problem. There are a few ways of doing so, as proposed by several researchers, e.g. see Hoek et al. 2002, Yang 2004, Sofianos 2006. In the case of a shallow tunnel, a fitting process proposed by Hoek et al 2002 is appropriate, in which the areas above and below the MC failure line are balanced with the HB failure line within the range of the tensile strength of the rock mass and the maximum confining stress ($\sigma_{\text{3max}}$) given by the Eq. (2):

$$\frac{\sigma'_{\text{3max}}}{\sigma'_{\text{cm}}} = 0.47 \left( \frac{\sigma'_{\text{cm}}}{\gamma H} \right)^{-0.94}$$  

Where $\sigma'_{\text{cm}}$ is the rock mass strength and a function of HB parameters, $\gamma$ is the unit weight of the rock mass and $H$ is the tunnel depth below the surface, here assumed as 30m which is approximately the depth of tunnel centre.

This fitting procedure results in the following equations for the equivalent MC strength parameters:

$$\phi' = \sin^{-1} \left[ \frac{6am_b(s + m_b\sigma'_{3n})^{a-1}}{2(1+a)(2+a) + 6am_b(s + m_b\sigma'_{3n})^{a-1}} \right]$$  

$$c' = \frac{\sigma_{ij}}{(1+a)(2+a)} \left[ (1+2a)s + (1-a)m_b\sigma'_{3n} \right] \left( s + m_i\sigma'_{3n} \right)^{a-1}$$

In which $m_b$, $\sigma'_{3n}$, $a$, $s$, can be calculated using HB parameters given in Table (1) by means of:

$$m_b = m_i \exp \left( \frac{GSI - 100}{28 - 14D} \right)$$  

$$s = \exp \left( \frac{GSI - 100}{9 - 3D} \right)$$  

$$a = \frac{1}{2} + \frac{1}{6} \left( e^{-GSI/15} - e^{-20/3} \right)$$  

$$\sigma'_{3n} = \frac{\sigma'_{\text{3max}}}{\sigma_{ij}}$$

In order to take the quality of tunnelling work and construction method into account, the disturbance factor (D) introduced by Hoek & Brown 1988 was considered. D is a factor that depends on the degree of disturbance of the rock mass subjected to blast damage and stress relaxation. It varies from 0 for undisturbed in-situ rock masses to 1 for very disturbed rock masses. Here it is assumed that the disturbance factor might change between undisturbed (D=0) to firmly disturbed (D=0.5).

Similar to the procedure mentioned in the foregoing section, two sets of strength parameters for the MC model can be derived using Eq. (3-4), and HB parameters given in Table (4), i.e. to compute the lower
bound of each set for \( c \) and \( \varphi \) the lower value of both \( D \) and GSI are used and for the upper bound the higher values are taken. The resulting random sets of MC strength parameters are summarized in Table (4).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Probability of assignment</th>
<th>( m_i )</th>
<th>( \sigma_{ci} ) (UCS)</th>
<th>( D )</th>
<th>GSI</th>
<th>( \varphi )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td>-</td>
<td>-</td>
<td>MPa</td>
<td>-</td>
<td>-</td>
<td>degree</td>
<td>kPa</td>
</tr>
<tr>
<td>Set No. 1</td>
<td>0.5</td>
<td>15</td>
<td>10</td>
<td>0-0.5</td>
<td>30-40</td>
<td>37-47</td>
<td>80-130</td>
</tr>
<tr>
<td>Set No. 2</td>
<td>0.5</td>
<td>25</td>
<td>50</td>
<td>0-0.5</td>
<td>30-40</td>
<td>53-60</td>
<td>160-280</td>
</tr>
</tbody>
</table>

4. Random Set Finite Element Method Procedure

In this section the procedure of a RS-FEM approach for typical geotechnical problems is elaborated by means of a real case study in tunnelling. This procedure includes the steps from preparing random sets of input parameters to retrieving typical system responses in terms of lower and upper probability measures from deterministic finite element computations. An overview of this procedure is schematically illustrated in Figure (2) and described in detail in the following:

**Step 1:** Selecting a calculation model to evaluate the behaviour of the geotechnical structure

Random Set theory is a general mathematical framework that can assess the uncertainty in a model irrespective of the model type (e.g. simple limit equilibrium analysis, analytical solution or numerical model) in which the behaviour of structural system is being evaluated. The benefits of numerical models such as Finite Element models are, especially in the case of very complex geotechnical structures (e.g. tunnelling), that both displacement and stress fields can be obtained employing advanced constitutive soil models, providing a better insight into the system behaviour than limit equilibrium analysis. Therefore, a finite element model is used for the work presented in this paper.

**Step 2:** Deciding on which input parameters are to be taken into account as random variables

It is common practice in underground projects, that due to the lack of primary information (e.g. small number of soil samples) or due to the inherent variability of parameters (e.g. seasonal fluctuation of the water table), designers consider different possibilities for subsurface conditions. In addition to material parameters, uncertainties associated to geometry and applied loads can also be handled (Fig. 2) within the RS-FEM approach. Secondly, the most influential parameters on the system response should be identified, which is not straightforward in a highly complex model, unless an appropriate sensitivity analysis is employed, e.g. as proposed by Peschl (2004). We can take the advantage of the application of sensitivity analysis to reduce the number of uncertain parameters, whose impact on the required result is negligible.

The number of all realisations required by the random set approach is given by:

\[
N_c = 2^N \prod_{i=1}^{N} n_i
\]  

(9)
Where \( N \) is the number of basic variables and \( n_i \), the number of information sources available for each variable. It means that the number of realizations goes up exponentially with increasing number of basic variables. Thus, it is worth performing such a sensitivity analysis to maintain the number of calculations as low as possible. Here, after a sensitivity analysis (Peschl 2004) four basic variables, namely the angle of internal friction \( \phi \), the effective cohesion \( c \), the stiffness \( E_s \) and the stress relaxation factor \( R_f \) which takes 3D effects of the construction process into account in the 2D model (e.g. Vogt et al. 1998). The random sets given in Table (5) are depicted in Figure (3) in terms of random set p-box (probability box).

**Step 3:** determination of spatial correlation length to consider spatial variability of soil parameters

Spatial variability of soil is a well-recognized phenomenon in geomechanics cited by many researchers e.g. Vanmarcke 1983, Griffiths and Fenton 2000, Lacasse and Nadim 1996. It is noteworthy that spatial variability can also be taken into account within the RS-FEM using the variance reduction technique proposed by Vanmarcke (1983) as shown by Schweiger and Peschl (2005). In this paper spatial correlation has been applied only on the modulus of elasticity, which changed both upper and lower bounds. It is acknowledged, however, that the Random Field Finite Element Method (RFEM) provides a more rigorous framework accounting for spatially random soil parameters and spatial correlation, e.g. Griffiths and Fenton (2004).

The final random set input variables utilized in further analyses are summarised in Table (5). It is noted that both sets chosen for the stress relaxation factor are based on expert’s opinion. In addition, the values for different stage constructions are correlated to each other, i.e. in any realisation the left or right extreme of \( R_f \) for top-heading, bench and invert are used simultaneously.

![Table 5. Input variables used in random set finite element analysis.](image)

**Step 4:** preparation of input files using random set model

After identification of input variables, the combinations of different sources and extremes of the parameters based on a random set model have to be calculated. Thus, data files for deterministic finite element calculations have to be provided and this may be performed with the additional pre-processing tool developed for this purpose.
Random Set Finite Element Method, Application to Tunnelling

Figure 2. Schematic RS-FEM procedure in typical geotechnical problems.
The concept of constructing a random set relation according to Tonon et al. 2000(a) & 2000(b) for the case considered here is as follows:
Let $u \in U$ be a vector of set value parameters, in which: $u = (E, c, \phi, R_f)$, and a random relation is defined on the Cartesian product $E \times C \times \Phi \times R$. As a result, according to combination calculus, the pairs generated by Cartesian product are given in the following vector:

$$E \times C \times \Phi \times R = \{(E_1, C_1, \Phi_1, R_1), (E_2, C_1, \Phi_1, R_1), (E_1, C_2, \Phi_1, R_1), \ldots, (E_2, C_2, \Phi_2, R_2)\}$$

(10)
Where the index of parameters denotes relevant set number and the index of pairs signifies one combination of basic variables. Because there are two sets for each basic variable, 16 combinations will be produced. For each combination, an interval analysis is required, by which the deterministic input parameters of the worst and the best case of each combination are being realised. As an example, deterministic input values of such analysis for the case of \((E_i, C_i, \Phi_i, R_i)\) are presented in Table (6).

<table>
<thead>
<tr>
<th>Run number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run mass probability</td>
<td>1</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
</tr>
<tr>
<td>Run mass probability</td>
<td>1</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>130</td>
<td>130</td>
<td>130</td>
<td>130</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>130</td>
<td>130</td>
<td>130</td>
<td>130</td>
<td>130</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>37</td>
<td>37</td>
<td>47</td>
<td>47</td>
<td>37</td>
<td>37</td>
<td>47</td>
<td>47</td>
<td>37</td>
<td>37</td>
<td>47</td>
<td>47</td>
<td>37</td>
<td>37</td>
<td>47</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>

*\(L\) denotes the lower extreme of a random set variable and \(U\) denotes the upper extreme.

Thereby, similarly the construction of 256 required realisations (see Eq. (9)) and relevant input files for a deterministic finite element analysis are accomplished. The next step is to determine the probability of assignment of each realisation. Assuming that our random variables are stochastically independent with reference to Tonon et al. 2000(b) section 2, the joint probability of the response focal element obtained through function \(f(u)\) (in this case, the finite element model) is the product of probability assignment \(m\) of input focal elements by each other. For instance, in the above case since mass probability of each set equals to 0.5 it gives:

\[
m(f(E_i, C_i, \Phi_i, R_i)) = m(E_i).m(C_i).m(\Phi_i).m(R_i) = 0.5 \times 0.5 \times 0.5 \times 0.5 = 0.0625
\]

**Step 5:** Performing all the deterministic finite element calculations and finding the bounds on system response

All the finite element calculations corresponding to random set input variables are performed. Results such as stresses, strains, displacements and internal forces in structural elements are obtained in terms of lower and upper bounds on discrete cumulative probability functions, which may be compared with measured data. In this section, some results of the tunnel problem are presented and some aspects of random set model are addressed.

The actual excavation sequences according to NATM principles leads to 7 calculation phases as follows:

1. Initial stresses
2. Pre-relaxation phase of top heading excavation
3. Installation of anchors and primary lining in top heading
4. Pre-relaxation phase of bench excavation
5. Installation of anchors and primary lining in bench
6. Pre-relaxation phase of invert excavation
7. Completion of primary lining and anchors

Figure 4. Position of points A and B.

The following typical results, which are of importance in a tunnel problem, are presented in terms of random set p-box depicted in Figure (5) showing the upper and lower cumulative probability of occurrence based on the random set model.

1. Vertical displacement of crown (Point A in Figure 4)
2. Vertical and horizontal displacement of side wall (Point B in Figure 4)
3. Maximum normal force in lining
4. Maximum moment in lining
5. Safety factor after top heading excavation without primary lining

To construct the Belief and Plausibility distribution function (i.e. lower and upper bounds) of a required response from deterministic FE calculations, the following procedure is pursued. Suppose that it is required to build up the p-box of tunnel crown displacement. The crown displacement values pertinent to all 16 realisations, given in Table (6), are sorted out to obtain the minimum and maximum of crown displacements which determine the focal element extremes of crown displacements corresponding to the combination \((E_1,C_1,\Phi_1,R_1)\). The displacement values of those realisations which are between the extremes are discarded. According to Eq. (11), the probability assignment of this focal element is 0.0625, which makes up one step in cumulative distribution function depicted in Figure (5). Similarly this process is repeated for other combinations e.g. \((E_1,C_2,\Phi_2,R_1)\)… to calculate the extremes of all focal elements of crown displacement. Finally, the left and right extremes of all focal elements are arranged in ascending order to obtain the upper and lower bounds for \(U_y-A\) respectively.
Figure (5) plots the p-box of vertical displacement of point A as well as both vertical and horizontal displacements of point B along with the respective measurements. This plot indicates that the results of the current approach are in good agreement with the measurements and therefore it shows its general capability to capture the uncertainties involved. In terms of discrete cumulative distributions, the measurement values illustrated in the plot are in two steps, since the measurement of only two sections with similar conditions to the model analysed are available. The more measurements available, the more steps appear in CDF of measurements. The measured values of $U_y$-A lie within the range specified as most likely values, while in the case of point B measured values are on the edge of the random set bounds. When the true system response falls outside the most likely values range it can be argued that either the models themselves are not appropriate (e.g. a simple homogenous Mohr-Coulomb model may not be sufficient to model a jointed rock mass) or the range of parameters is not representative of the subsurface conditions. Generally, the most likely values are defined as values with the highest probability of occurrence, where the slope of the corresponding cumulative distribution function is steepest. For the purpose of illustration, it is assumed that the most likely values have a probability of 50% as shown in Figure (5). In addition, the mean value of the true system response obtained by random set bounds should be within the following range given by Tonon et al. 2000(a):

$$\mu = \frac{1}{N} \sum_{i=1}^{N} m_i \cdot \inf(A_i); \frac{1}{N} \sum_{i=1}^{N} m_i \cdot \sup(A_i)$$ (12)

Where $\inf(A)$ and $\sup(A)$ denote lower and upper extreme of focal element A respectively. The intervals obtained from both the most likely range definition and those calculated from Eq. (12) indicate a good conformity and they have been tabulated below (Table 7).

<table>
<thead>
<tr>
<th>Interval of true mean values [Eq. 12]</th>
<th>Uy-A [mm]</th>
<th>Uy-B [mm]</th>
<th>Ux-B [mm]</th>
<th>FOS Top-heading</th>
<th>Max Moment [kN.m/m]</th>
<th>Max Normal Force [kN/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower value</td>
<td>1.7</td>
<td>1.1</td>
<td>0.5</td>
<td>2.45</td>
<td>4.05</td>
<td>190</td>
</tr>
<tr>
<td>upper value</td>
<td>4.2</td>
<td>2.5</td>
<td>1.1</td>
<td>3.7</td>
<td>14.2</td>
<td>487</td>
</tr>
</tbody>
</table>

If one argues that the range of internal forces obtained is too wide for design purposes, it should be realized that this is a consequence of the wide range of imprecise input model parameters. The random set model just propagates the uncertainties in the state of our current knowledge through the model and nothing more. Wide range results imply that it is necessary to decrease the imprecise uncertainties by means of performing additional site investigation schemes or utilising other measures to achieve more reliable sources of information with a narrower range.
Figure 5. Considered results in the tunnel problem in terms of random set p-box.
According to Tonon et al. 2000(b), providing that the image function $f(u)$ as well as its derivatives are continuous and $f(u)$ is strictly monotonic with respect to each input variable $u$, the image function can be evaluated only twice for each focal element, which dramatically reduces the number of realisations. In a very complex numerical model, the second condition hardly holds as follows from the results of the interval analysis relating to one of the random combinations $(E_1,C_1,\Phi_1,R_1)$ depicted in Figure (6). This is in particular the case when different types of system responses, e.g. internal force, displacements and safety factor of a stage construction are of interest.

As it can be seen from Figure (6), both combinations LLLL and UUUU of input parameters result in the maximum and minimum of crown displacement respectively, while in the case of maximum moment in the lining, combination LLLL and UUUU result in the extremes of the focal element. It means that for different desired responses, different combinations might produce a worse and best case. As depicted in Figure (6), finding a trend in different system responses is very difficult in advance. It suggests that all $2^N$ vertices for each focal element of response should be evaluated in a complicated numerical model to find out the worst and best cases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The results of interval analysis relating to combination $(E_1,C_1,\Phi_1,R_1)$.}
\end{figure}

**Step 6:** Reliability analysis using Random Set results

In practice, a tunnel design is usually accomplished based on a deterministic analysis which can be considered as one of the possibilities of all results within two probable bounds achieved by means of a random set analysis. In some cases, this range of results from RS-FEM might be very wide and therefore the question arises: “To what extent is the tunnel design based on deterministic analysis reliable?” In this section an answer to this question using the results of random set analysis is attempted. To do so a suitable performance function has to be defined. A serviceability or ultimate limit state function relating to critical
deformations (e.g. tunnel crown displacement) and/or maximum allowable stresses in the lining identifying a criterion to avoid serious cracks would be a possibility for the problem under consideration.

To demonstrate the applicability of RSM in reliability analysis two serviceability limit state functions relating to deformation and design of the shotcrete lining are considered.

1. First, the serviceability limit state of the shotcrete lining is addressed. It is based on damage of the lining due to cracking after the tensile capacity of the material is exceeded. The admissible value of the normal force $N_{lim}$ as a function of the eccentricity $e(x)$ as given by Schikora & Ostermeier (1988) is:

$$N_{lim} = \frac{f_c d}{F_s} \left(1 - 2 \frac{e(x) + e_a}{F_s d}\right)$$

Where $f_c$ = uniaxial strength of shotcrete; $e_a$ = imperfection; $d$ = thickness of lining; $e(x)$ = eccentricity $M/N$; $M$ = bending moment; $N$ = axial force; $F_s$ = factor of safety. Thus, the serviceability limit state function which should be evaluated is defined in Eq. (14).

$$g(x) = N_{lim} - N$$

It is required to achieve a continuous CDF of results using best-fit method. For this purpose, commercial software such as the package @RISK (V5 2007) may be employed. As depicted in Figure (7), two different families of distributions (Normal and Beta distribution) were tried to investigate their impact on probability of unsatisfactory performance. Following this step, a Monte Carlo simulation was executed using Latin Hypercube sampling technique and employing aforementioned software to carry out a reliability analysis for serviceability conditions of the shotcrete lining at the final stage of construction. The bounds of $g(x)$ portrayed in Figure (7) are derived directly from random set analysis similar to internal forces of the lining, that is, $e(x)$ which is a function of moment and normal force, is the only variable of $g(x)$. The following assumption were made regarding other variables involved in Eq. (13), i.e. for the shotcrete a C20/25 with an uniaxial strength of about 17.5MPa is used and to cover imperfections an eccentricity of $e_a = 2.0$cm and for the serviceability limit state a safety factor of $F_s = 2.1$ is considered. The lower and upper values of probability of exceeding the admissible normal force in the lining, $N > N_{lim}$, where cracking takes place, corresponding to lower and upper probability of $g(x)$ with both Normal and Beta distributions are approximately zero. In this special case different distributions had no effect on the reliability results since the performance function is far away from failure. The value of probability of failure indicates that the shotcrete satisfies the serviceability criterion and it is expected that major cracking will not occur in the lining.

2. Secondly a performance function relating to tunnel crown deformation is considered. It is common practice in tunnelling to constrain tunnel convergence to a certain limit to provide the required space for operation. The simplest form of such a performance function is assumed below.

$$g(U_{yd}) = 6 - U_{yd}$$

Where $U_{yd}$ is vertical displacement of tunnel crown downwards, 6 is a trigger value as a criterion adopted in this example. Similar to the previous example a continuous CDF fitted to random set results depicted in Figure (7) were used to obtain the probability of failure given by Eq. (16).

$$p_f = p(g(x) \leq 0) = \int_{g(x) \leq 0} f_x(x) dx$$

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The lower and upper values of probability of ‘failure’ (meaning in this context that the trigger value for crown displacement is exceeded) calculated by means of Monte-Carlo simulation are 0.00001 and 0.032 respectively.

\[ U_y - A \ [\text{mm}] \]

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\[ g(x) \ [\text{kN/m}] \]

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**Figure 7.** Serviceability limit state function of the tunnel crown displacement and primary lining in terms of random set p-box.

### 5. Conclusion

A real tunnel project was used to demonstrate the validity of RS-FEM framework in geomechanical problems where engineers have to face not only imprecise data but also lack of information. From a practical point of view, RS-FEM proved its capability of capturing uncertainty in a complex geotechnical application. Presenting results in terms of probability measures in bounds based on imprecise input data are useful in context of the observational method, which is generally applied in NATM tunnelling. Comparison of analysis and in situ behaviour showed that in some results the measurements are on the margin of random set bounds. If they would be outside the bounds, probably it implies that either the assumptions made in the numerical model (e.g. homogeneous material in a continuum) is not able to fully characterise the rock mass behaviour or the input sources are not reliable. This can be viewed as an advantage of the current approach because comparison of the results with measurements indicates whether overall assumptions made for the model and the parameters have been reasonable, making it easier for the designers to establish criteria for revising modelling assumptions.
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References


Random Set Finite Element Method, Application to Tunnelling

