Confidence single-level formulation in structural robust optimization and its solution aspect

Xu Guo
State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian, 116023, P.R. China, guoxu@dlut.edu.cn

Abstract: Structural robust optimization problems are often solved via the so-called Bi-level approach. The solution of Bi-level optimization problems often involves large computational efforts and sometimes the convergence behavior is not so good because of the non-smooth nature of the Bi-level formulation. In the present paper, confidence single-level nonlinear semi-definite programming (NLSDP) formulations for structural robust optimization problems under stiffness uncertainties are proposed. This is achieved by using some technical tools such as S-procedure and quadratic embedding in convex analysis. The resulting NLSDP problems are then solved using Augmented Lagrange Multiplier Method with sound mathematical properties. Furthermore, the deficiencies of the naive single level formulation in literatures are also analyzed. Numerical examples show that confidence robust optimal solutions can be obtained with the proposed approach effectively.

Keywords: Robust design and optimization; Parameter uncertainty; Nonlinear semi-definite program; Confidence robust optimal solution

1. Introduction

Traditional optimal structural design is always performed based on the deterministic optimization model in which parameters such as material properties, applied loads and geometry coordinates are believed to have definite values. However, uncertainties of these parameters are unavoidable in real-world applications and it is well known that solutions to optimization problems can exhibit remarkable sensitivity to parameter perturbations.

A lot of optimization methods that take the parameter uncertainties into consideration have been proposed in the last two decades. Generally speaking, these methods are mainly based on two kinds of uncertainty models: probabilistic and non-probabilistic models. In probabilistic based model, the uncertainty of the parameters is described by their probabilistic distribution functions. The corresponding optimization problems generally aim at minimizing the cost of the design under a prescribed failure probability [Royset, Kiureghian and Polak (2001); Choi, Tu and Park (2001); Jung and Lee (2002); Papadrakakis and Lagaros (2002); Kharmanda, Olhoff and Lemaire (2004)] (often named as Reliability-Based Optimization-RBO) or minimizing the mean value of the objective function and the variance of the structural responses simultaneously [Lee and Park (2001); Sandgren and Cameron (2002); Lee, Eom, Park and Lee (1996); Lee and Park (2002)](often named as Robust Design and Optimization-RDO). Although probabilistic modeling provides an attractive framework for robust optimal design, it is worth noting that, sometimes, it relies too much on the accurate probabilistic information of the parameter uncertainties. It has been shown that when the probability information is inaccurate, large errors can be introduced in the calculation of the failure probabilities [Elishakoff (1995)].

4th International Workshop on Reliable Engineering Computing (REC 2010)
Edited by Michael Beer, Rafi L. Muhanna and Robert L. Mullen

ISBN: 978-981-08-5118-7. *Published by* Research Publishing Services.
doi:10.3850/978-981-08-5118-7_061

In contrast to the probabilistic uncertainty model, non-probabilistic modeling is often used when the knowledge about the uncertainty is limited. In this framework, it is only assumed that the uncertain parameters belong to an unknown-but-bounded set and there is no need to obtain the statistical distributions of the uncertainty parameters. Robust optimization under this uncertainty framework generally aims at finding a solution which is optimal for any realization of the uncertainty in a given set and also named as Worst Case Design and Optimization (WCDO). The ultimate goal of WCDO is to minimize the cost of the design while ensuring the safety of structure under the worst combination of uncertain parameters. It is worth noting that although no statistical information of uncertainty is required in non-probabilistic modeling based robust optimization approaches, how to choose the sizes of the bounded set is an important issue. It is obvious that if the size of the uncertainty sets is too large, the corresponding optimal designs may be too conservative. For research works under the WCDO framework, we refer the reader to [Ben-Haim and Elishakoff (1990); Elishakoff, Haftka and Fang (1994); Pantelides and Ganzerli (1998); Lombardi and Haftka (1998); Au, Cheng, Tham and Zeng (2003); Gurav, Goosen and VanKeulen (2005); Jiang, Han and Liu (2007)]. In the present paper, robust optimization of truss structures will also be discussed in the non-probabilistic uncertainty framework.

In general, a WCDO problem can be formulated as the following single-level semi-infinite optimization program

$$\begin{aligned} &\text{find} \quad \boldsymbol{x} = (x_1, \dots, x_n)^{\top} \in \mathbf{R}^n \\ &\quad \min \quad f(\boldsymbol{x}) \\ &\text{s.t.} \quad g_j(\boldsymbol{x};\boldsymbol{p}) \leq 0, \forall \boldsymbol{p} \in \mathrm{U}_{\boldsymbol{p}},\ j = 1, \dots, m, \\ &\qquad x_i^l \leq x_i \leq x_i^u,\ i = 1, \dots, n. \end{aligned} \tag{1.1}$$

where $f: \mathbf{R}^n \to \mathrm{R}$ and $g_j : \mathbf{R}^n \times \mathbf{R}^k \to \mathrm{R}$ are assumed to be continuous differentiable functions in their domains, $\boldsymbol{x} \in \mathbf{R}^n$ is the vector of design variables and $\boldsymbol{p} \in \mathbf{R}^k$ is the vector of uncertain parameters. The uncertainty set is denoted by $\mathrm{U}_{\boldsymbol{p}} \subset \mathbf{R}^k$, which is supposed to be closed in this paper. x_i^l and x_i^u denote the lower and upper bounds of x_i, respectively. The goal of (1.1) is to minimize the total cost among all the feasible candidate designs for all realization of the parameters in $\mathrm{U}_{\boldsymbol{p}}$. It is worth noting that (1.1) is a mathematical program with infinite number of constraints since the uncertainty set $\mathrm{U}_{\boldsymbol{p}}$ is in general a continuous set with uncountable infinite number of elements.

In order to avoid dealing with infinite number of constraints, one possible way is to replace (1.1) by the following equivalent problem formulation

$$\begin{aligned} &\text{find} \quad \boldsymbol{x} = (x_1, \dots, x_n)^{\top} \in \mathbf{R}^n \\ &\quad \min \quad f(\boldsymbol{x}) \\ &\text{s.t.} \quad \max_{\boldsymbol{p} \in \mathrm{U}_{\boldsymbol{p}}} g_j(\boldsymbol{x};\boldsymbol{p}) \leq 0,\ j = 1, \dots, m, \\ &\qquad x_i^l \leq x_i \leq x_i^u,\ i = 1, \dots, n. \end{aligned} \tag{1.2}$$

Equation (1.2) represents a nested Bi-level program. In the upper-level problem, the aim is to find the best design by optimal selection of design variables, while in the lower-level problem, optimization is carried out to find the worst case structural responses which are then used to examine the feasibility of a given design.

Although formulating the problem in a Bi-level form is a common practice in structural robust optimization, it, however, still has some problems which deserve further explorations. The first one is the global optimality of the lower-level problem in (1.2). The second problem of the Bi-level program formulation is associated with its numerical solution aspect. In order to overcome the above difficulties, a possible choice is to formulate the problem as a single-level program [Liang and Mourelatos (2007); McDonald and Mahadevan (2008); Kuschel and Rackwitz (1997); Agarwal and Renaud (2004)]. The advantage of the single-level formulation is two-fold: First, compared to the Bi-level formulation, the computation efforts can be reduced greatly since only one-loop optimization problem is necessary to be solved. Second, if correctly formulated, the requirement of the global optimality of the lower-level program can be circumvented. This is very important for the confidence solution of the robust optimization problems.

In order to maximize the robust function of truss structures under load uncertainties, Kanno and Takewaki [Kanno and Takewaki (2006)] formulated the problem as a nonlinear semi-definite programming program and developed a sequential linear semi-definite programming approach to solve it. To the best of our knowledge, this is the first confidence single-level formulation for stress/displacement-constrained structural robust optimization under load uncertainties.

In the present paper, the construction of confidence single-level formulations for structural robust design under stiffness uncertainties and its solution aspect are discussed. By using the quadratic embedding technique of uncertainty and the S-procedure in convex analysis, we reformulate the original semi-infinite problem as a single-level nonlinear semi-definite programming (NLSDP) optimization problem. Truss structures are used as test-bed to illustrate the ideas. The rest of this paper is organized as follows: Confidence single-level formulations for robust design of truss structures for static and steady state dynamic cases will be presented in Section 2 and Section 3, respectively. An Augmented Lagrange Multiplier Method, which will be used to solve the resulting NLSDP problems, is introduced in Section 4. The proposed solution procedure is then applied to solve several numerical examples in Section 5 for demonstration of its effectiveness. Finally, some concluding remarks are given in Section 6.

2. Confidence single-level formulation of robust design under static condition

The problem considered in this section is to find the optimal truss structure with minimum total weight taking the static structural performance constraints and the possible stiffness uncertainties into considerations. Here we assume that the stiffness uncertainty results from the manufacture error or the material degradation of the cross sectional area of the truss bar. The nominal values of the cross sectional areas of the bars are taken as design variables. In the form of semi-infinite program, the optimization problem can be written as follows

$$\begin{aligned} &\text{find} \quad \tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_{n^m})^\top \in \mathbf{R}^{n^m} \\ &\min \quad W = \sum_{i=1}^{n^m} \rho_i l_i \tilde{a}_i \\ &\text{s.t.} \quad \mathcal{U}_j \subset [g_j^l, g_j^u], j = 1, \dots, m, \\ &0 < \tilde{a}_i^l \le \tilde{a}_i, \ i = 1, \dots, n^m. \end{aligned} \tag{2.1}$$

In (2.1), n^m denotes the total number of bars in the structure. m denotes the number of the structural performance constraints. ρ_i and l_i are the mass density and the length of the i-th bar, respectively. $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_{n^m})^\top \in \mathbf{R}^{n^m}$ is the vector of design variables with $\tilde{a}_i$ denoting the nominal value of the cross sectional area of the i-th bar. $\tilde{a}_i^l$ denotes the lower bound of $\tilde{a}_i$. $\mathrm{U}_{\boldsymbol{a}}$ denotes the unknown-but-bounded set covering all possible values of $\boldsymbol{a} \in \mathbf{R}^{n^m}$.

$$\mathcal{U}_j = \{\boldsymbol{d}_j^\top \boldsymbol{u} | \boldsymbol{K}(\boldsymbol{a})\boldsymbol{u} = \boldsymbol{f}, \forall \boldsymbol{a} \in \mathrm{U}_{\boldsymbol{a}}\} \tag{2.2}$$

is the attainable set of the concerned structural response when $\boldsymbol{a}$ are varied in $\mathrm{U}_{\boldsymbol{a}}$. $\boldsymbol{f}$ in Eq. (2.2) is the applied load vector. $\boldsymbol{d}_j \in \mathbf{R}^{n^d}$ (n^d is the number of degrees of freedom) is a constant vector which relates the j-th concerned structural response (e.g. nodal displacement component or stress of the bar) to the nodal displacement vector $\boldsymbol{u}$. $\boldsymbol{K}(\boldsymbol{a}) = \sum_{i=1}^{n^m}(E_i a_i/l_i)\,\boldsymbol{r}_i\boldsymbol{r}_i^\top$ is the global stiffness matrix of the structure with E_i, l_i and $\boldsymbol{r}_i \in \mathbf{R}^{n^d}$ denoting the Young's modulus, the length and the vector of direction cosines of the i-th bar. g_j^l and g_j^u are the lower and upper bounds of the j-th structural behavior constraint, respectively. In the present paper, it is assumed that the uncertainties of $\boldsymbol{a}$ are described by the following interval model

$$\mathrm{U}_{\boldsymbol{a}} = \mathrm{U}_{\boldsymbol{a}}(\tilde{\boldsymbol{a}};\boldsymbol{\eta}) = \{\boldsymbol{a}|(1-\eta_i)\tilde{a}_i \le a_i \le (1-\eta_i)\tilde{a}_i, 0 \le \eta_i < 1, i = 1,\dots,n^m\}. \quad (2.3)$$

In Eq. (2.3) $\eta_i \in [0,1)$ represents the ratio of the magnitude of the perturbation of a_i around its nominal value $\tilde{a}_i$. The motivation for the above choice of the representation of the uncertainties arises from the fact that there are generally no correlations among the stiffness uncertainties of the truss bars.

Optimization problem (2.1) is a semi-infinite program since the constraints $\mathcal{U}_j \subset [g_j^l, g_j^u],\ j = 1,\dots,m$ should be satisfied for all possible realizations of $\boldsymbol{a} \in \mathrm{U}_{\boldsymbol{a}} = \mathrm{U}_{\boldsymbol{a}}(\tilde{\boldsymbol{a}};\boldsymbol{\eta})$.

To make the problem more computationally tractable, (2.1) is usually reformulated in the following Bi-level form

$$\begin{aligned}
&\text{find} \quad \tilde{\boldsymbol{a}} = (\tilde{a}_1,\dots,\tilde{a}_{n^m})^\top \in \mathbf{R}^{n^m} \\
&\min \quad W = \sum_{i=1}^{n^m} \rho_i l_i \tilde{a}_i \\
&\text{s.t.} \max_{\substack{\boldsymbol{K}(\boldsymbol{a})\boldsymbol{u}=\boldsymbol{f} \\ \forall \boldsymbol{a}\in \mathrm{U}_{\boldsymbol{a}}}} \boldsymbol{d}_j^\top \boldsymbol{u} - g_j^u \le 0, j = 1,\dots,m, \\
&\quad - \max_{\substack{\boldsymbol{K}(\boldsymbol{a})\boldsymbol{u}=\boldsymbol{f} \\ \forall \boldsymbol{a}\in \mathrm{U}_{\boldsymbol{a}}}} \boldsymbol{d}_j^\top \boldsymbol{u} + g_j^l \le 0, j = 1,\dots,m, \\
&\quad 0 < \tilde{a}_i^l \le \tilde{a}_i,\ i = 1,\dots,n^m.
\end{aligned} \quad (2.4)$$

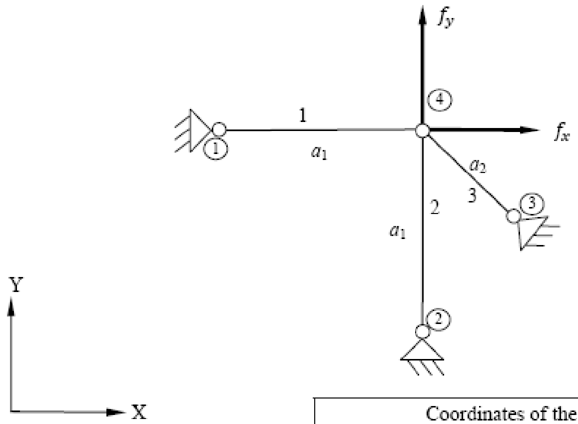
The optimization problem $\max_{\substack{\boldsymbol{K}(\boldsymbol{a})\boldsymbol{u}=\boldsymbol{f} \\ \forall \boldsymbol{a}\in \mathrm{U}_{\boldsymbol{a}}}} \boldsymbol{d}_j^\top \boldsymbol{u} - g_j^u$ in Eq. (2.4) is called as the lower-level problem, which is used to determine the feasibility of a design variable transferred from the upper-level total weight minimization problem. The advantage of Eq. (2.5) is that the number of constraints is finite.

Let us consider the robust optimization problem of a truss structure shown in Fig. 1. For simplicity, suppose that the Young's moduli and the material densities of the bars are $E_i = 1.0, i = 1,2,3$ and $\rho_i = 1.0, i = 1,2,3$, respectively. The applied load vector is $\boldsymbol{f} = (f_1, f_2)^\top$. Since $l_1 = l_2 = 1.0$ and $l_3 = 1/2$, the system equilibrium equations can be written as

$$\begin{pmatrix} a_1 + a_2 & -a_2 \\ -a_2 & a_1 + a_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad (2.5)$$

and the horizontal displacement at the free node can be solved as

$$u_1 = \frac{f_1 a_1 + (f_1 + f_2)a_2}{a_1^2 + 2a_1 a_2}\ . \quad (2.6)$$



Coordinates of the nodes			
Node-1	Node-2	Node-3	Node-4
$(-1,0)$	$(0,-1)$	$(\sqrt{2}/4, -\sqrt{2}/4)$	$(0,-1)$

Figure.1 A 3-bar truss structure

Now let $a_2 \equiv 1$ and $\boldsymbol{f} = \left(1, -\sqrt{2.5}\right)^{\mathrm{T}}$. Assuming that the nominal values of the cross sectional area of the bars are $\tilde{a}_1 = 2$ and a_1 might experience not more than 50 percent ($\eta_1 = 0.5$) stiffness degradation around its nominal value, then it can be verified that u_1 have one global maximum $u_1^{\max} = 0.17820$ at $a_1 = 1.8059$ and two local optima (one is at $a_1 = 1$ and the other is at $a_1 = 3$) in the uncertainty interval of a_1 as shown in Fig. 2.

Now let us consider the structural response constraint such that the perturbation of u_1 around its nominal value $\tilde{u}_1$ should satisfy

$$\max_{1 \le a_1 \le 3} u_1(a_1, 1) - \tilde{u}_1 \le 0.01694, \tag{2.7}$$

$$\tilde{u}_1 - \min_{1 \le a_1 \le 3} u_1(a_1, 1) \le 0.01694. \tag{2.8}$$

Under this circumstance, it can be seen from Fig. 2 that the minimization problem in (2.8) has two local optima 'A' and 'D'. At local optimum 'A', constraint in (2.8) is satisfied while at global optimum 'D', the constraint is violated. Therefore, if the optimization process for finding the worst case structural response terminates at the local optimum 'A' (Converging to 'A' is highly possible if local optimization methods are used and this has been confirmed by numerical experiments), then the current design $\tilde{a}_1 = 2$ may be erroneously evaluated as a feasible point. But it is in fact an infeasible point since at point 'A'

$$|u_1(a_1 = \tilde{a}_1 - 1, 1) - u_1(a_1 = \tilde{a}_1 = 2, 1)| = |0.13962 - 0.17736| > 0.01694.$$

This example demonstrates clearly the importance of solving the lower-level program in WCDO problems with global optimality. If the robust optimization problem (minimum total weight robust design) is formulated as

$$
\begin{aligned}
\text{find} \quad & \tilde{a}_1 \in \mathrm{R} \\
& \min \quad W = 2\tilde{a}_1 + \frac{1}{2} \\
\text{s.t.} \quad & \max_{0.5\tilde{a}_1 \le a_1 \le 1.5\tilde{a}_1} u_1(a_1, 1) - \tilde{u}_1 \le 0.01694, \\
& \tilde{u}_1 - \min_{0.5\tilde{a}_1 \le a_1 \le 1.5\tilde{a}_1} u_1(a_1, 1) \le 0.01694, \\
& \tilde{a}_1 \ge 2.
\end{aligned}
\tag{2.9}
$$

where $\tilde{u}_1 = u_1(a_1 = \tilde{a}_1, 1)$. Then it is obviously that $\tilde{a}_1 = 2$ might be evaluated erroneously as an optimal solution (in a fact an infeasible solution) of (2.9) if it is solved with traditional approaches.

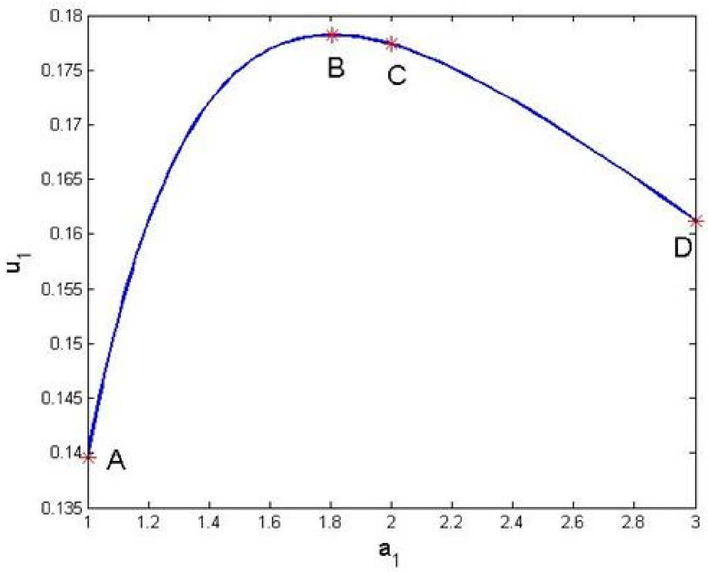


Figure. 2 Local and global optima of u_1

In order to resolve the above mentioned problem, one possible way is to formulate the problem (in a proper way) as a single-level program. With use of this formulation, the difficulty associated with the global optimality of the lower-level program can be circumvented. In the following, technical tools from convex analysis will be used to achieve this goal.

2.1 REPRESENTATION OF STRUCTURAL BEHAVIOR CONSTRAINTS IN QUADRATIC FORM

As shown in [Kanno and Takewaki (2006)], for optimal design of truss structures, the structural behavior constraints can always be written as the quadratic inequalities in term of the nodal displacement vector $\boldsymbol{u}(\boldsymbol{a}) \in \mathbf{R}^{n^d}$ in the following form

$$
\mathfrak{P} = \left\{ \boldsymbol{u} \in \mathbf{R}^{n^d} \middle| \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^{\top} \boldsymbol{\Phi}_l \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \ge 0, l = 1, \dots, m \right\}, \tag{2.10}
$$

where $\boldsymbol{\Phi}_l \in \mathbf{S}^{n^d+1}$ (the space of $(n^d + 1) \times (n^d + 1)$ symmetric matrix).

2.2 ATTAINABLE SET OF CONCERNED STRUCTURAL RESPONSE UNDER PRESCRIBED INTERVAL TYPE STIFFNESS UNCERTAINTIES

In this subsection, the attainable set of concerned structural response will be constructed. Herein the so-called attainable set is the set that the values of the structure response can be achieved for all realizations of the uncertainty in a prescribed set. With use of the Sherman-Morrison-Woodbury formula in matrix analysis, the displacement vector around its nominal value can be expressed as a function of $\Delta\boldsymbol{a}$ in the following form

$$\boldsymbol{u}(\tilde{\boldsymbol{a}}+\Delta\boldsymbol{a}) = \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{K}}^{-1}\boldsymbol{V}\boldsymbol{p}(\Delta\boldsymbol{a}), \tag{2.14}$$

where $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{K}}^{-1}\boldsymbol{f} = \boldsymbol{K}(\tilde{\boldsymbol{a}})^{-1}\boldsymbol{f}$ is the nominal displacement vector. $\boldsymbol{V} = (\boldsymbol{r}_1, \dots, \boldsymbol{r}_{n^m}) \in \mathbf{R}^{n^d \times n^m}$,

$$\boldsymbol{p}(\Delta\boldsymbol{a}) = \left(\boldsymbol{I} + \boldsymbol{H}(\Delta\boldsymbol{a})\boldsymbol{V}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{V}\right)^{-1}\boldsymbol{H}(\Delta\boldsymbol{a})\boldsymbol{V}^\top\tilde{\boldsymbol{u}} \tag{2.15}$$

and

$$\begin{aligned}\boldsymbol{H}(\Delta\boldsymbol{a}) &= \mathrm{diag}\left(\frac{\Delta a_1 E_1}{l_1}, \dots, \frac{\Delta a_{n^m} E_{n^m}}{l_{n^m}}\right)\\ &= \mathrm{diag}\left(\frac{\eta_1\zeta_1\tilde{a}_1E_1}{l_1}, \dots, \frac{\eta_{n^m}\zeta_{n^m}\tilde{a}_{n^m}E_{n^m}}{l_{n^m}}\right) \in \mathbf{S}^{n^m},\end{aligned} \tag{2.16}$$

with $|\zeta_i| \le 1,\ \ i = 1, \dots, n^m$.

From Eq. (2.14), it is known that the attainable set of $\boldsymbol{u}$ is solely determined by the attainable set of $\boldsymbol{p}$ satisfying

$$\left(\frac{\eta_i\tilde{a}_iE_i}{l_i}\right)^2 \left(\boldsymbol{p}^\top\boldsymbol{b}_i\boldsymbol{b}_i^\top\boldsymbol{p} - 2e_i\boldsymbol{b}_i^\top\boldsymbol{p} + e_i^2\right) \ge \boldsymbol{p}^\top\boldsymbol{D}_i\boldsymbol{p}, i = 1, \dots, n^m. \tag{2.17}$$

Defining $\boldsymbol{\xi} = (\boldsymbol{p}^\top, 1)^\top \in \mathbf{R}^{n^m+1}$,

$$\alpha_i = \left(\frac{\eta_i\tilde{a}_iE_i}{l_i}\right)^2, \boldsymbol{b}_i = (B_{i1}, \dots, B_{in^m})^\top,\ i = 1, \dots, n^m, \tag{2.18}$$

then Eq. (2.17) can be expressed in the following compact form

$$\boldsymbol{\xi}^\top\boldsymbol{\Omega}_i\boldsymbol{\xi} \ge 0, \forall i = 1, \dots, n^m, \tag{2.19}$$

where

$$\boldsymbol{\Omega}_i = \boldsymbol{\Omega}_i(\tilde{\boldsymbol{a}}, \eta_i) = \begin{pmatrix} \alpha_i\boldsymbol{b}_i\boldsymbol{b}_i^\top - \boldsymbol{D}_i & -\alpha_ie_i\boldsymbol{b}_i \\ \text{sym.} & \alpha_i(e_i)^2 \end{pmatrix} \in \mathbf{S}^{n^m+1}. \tag{2.20}$$

Therefore the attainable set $\mathfrak{D}_p(\tilde{\boldsymbol{a}}, \boldsymbol{\eta})$ of $\boldsymbol{p}$ can be obtained as

$$\mathfrak{D}_p(\tilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \left\{\boldsymbol{p} \in \mathbf{R}^{n^m} \middle| \boldsymbol{\xi}^\top\boldsymbol{\Omega}_i(\tilde{\boldsymbol{a}}, \eta_i)\boldsymbol{\xi} \ge 0, \forall i = 1, \dots, n^m\right\}, \tag{2.21}$$

The admissible set associated with the concerned structural behavior constraints can also be expressed by the quadratic inequalities in terms of $\boldsymbol{p}$. For example, if the concerned nodal displacement constraint is

$$\|\boldsymbol{u}_c\|^2 = \|\boldsymbol{R}_c\boldsymbol{u}(\boldsymbol{a})\|^2 \le \bar{u}_c \tag{2.22}$$

(where $\boldsymbol{R}_c \in \mathbf{R}^{q \times n^d}$ is a localization matrix), then with use of Eq. (2.14), it can be expressed in the following equivalent form

$$\left(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{K}}^{-1}\boldsymbol{V}\boldsymbol{p}(\Delta\boldsymbol{a})\right)^{\top} \boldsymbol{R}_c^{\top}\boldsymbol{R}_c \left(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{K}}^{-1}\boldsymbol{V}\boldsymbol{p}(\Delta\boldsymbol{a})\right) \le \bar{u}_c. \tag{2.23}$$

Hence the admissible set associated with Eq. (2.22) can be expressed in terms of $\boldsymbol{p}$ as follows

$$\mathfrak{P} = \mathfrak{P}(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c) = \left\{\boldsymbol{p} \in \mathbf{R}^{n^m} \middle| -\boldsymbol{\xi}^{\top}\boldsymbol{\Psi}_i(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c)\boldsymbol{\xi} \ge 0\right\}, \tag{2.24}$$

where

$$\boldsymbol{\Psi}_i(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c) = \begin{pmatrix} \boldsymbol{V}^{\top}\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^{\top}\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1}\boldsymbol{V} & -\boldsymbol{V}^{\top}\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^{\top}\boldsymbol{R}_c\tilde{\boldsymbol{u}} \\ \text{sym.} & \tilde{\boldsymbol{u}}^{\top}\boldsymbol{R}_c^{\top}\boldsymbol{R}_c\tilde{\boldsymbol{u}} - \bar{u}_c \end{pmatrix} \in \mathbf{S}^{n^m+1} \tag{2.25}$$

and $\boldsymbol{\xi} = (\boldsymbol{p}^{\top}, 1)^{\top} \in \mathbf{R}^{n^m+1}$. Other kinds of structural behavior constraints (e.g. stress constraints) can also be dealt with in a similar way.

2.3 CONFIDENCE SINGLE-LEVEL FORMULATION OF ROBUST DESIGN UNDER STATIC CONDITION

Armed with the results developed in the previous sections, we are now at the position to establish the single level formulation for the considered problem. The key issue for constructing single level formulation is to replace the original program with infinite number of constraints in (2.1), which is computationally intractable, by a program with finite number of constraints. This goal can be achieved with the help of the aforementioned technical lemmas.

As shown in the previous subsection, the structural behavior constraints (e.g. $\|\boldsymbol{u}_c\|^2 \le \bar{u}_c$) can always be expressed in the following form

$$\mathfrak{P}_c = \mathfrak{P}_c(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c) = \left\{\boldsymbol{p} \in \mathbf{R}^{n^m} \middle| -\boldsymbol{\xi}^{\top}\boldsymbol{\Psi}(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c)\boldsymbol{\xi} \ge 0\right\}.$$

Recalling the fact that when the stiffnesses are varied in the prescribed uncertain interval, the attainable set of the structural response is

$$\mathfrak{D}_p(\tilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \left\{\boldsymbol{p} \in \mathbf{R}^{n^m} \middle| \boldsymbol{\xi}^{\top}\boldsymbol{\Omega}_i(\tilde{\boldsymbol{a}}, \eta_i)\boldsymbol{\xi} \ge 0, \forall i = 1, \dots, n^m\right\}.$$

Then the robustness of design $\boldsymbol{a} = \tilde{\boldsymbol{a}}$ requires that

$$\forall \boldsymbol{p} \in \mathfrak{D}_p(\tilde{\boldsymbol{a}}, \boldsymbol{\eta}) \Rightarrow \boldsymbol{p} \in \mathfrak{P}_c(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c), \tag{2.26}$$

See Fig.3 for reference.

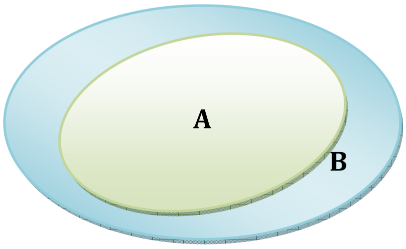
According to Lemma 2.1 and 2.2, the implication in Eq. (2.26) holds if there exist $\tau_1, \tau_2, \dots, \tau_{n^m} \ge 0$ such that

$$-\boldsymbol{\Psi}(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c) - \sum_{i=1}^{n^m} \tau_i \boldsymbol{\Omega}_i(\tilde{\boldsymbol{a}}, \eta_i) \succcurlyeq \mathbf{O}. \tag{2.27}$$

Based on the above results, the original semi-infinite program in (2.1) can be reformulated into a conservative single-level program in the following form

$$\begin{aligned} &\text{find} \quad \tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_{n^m})^{\top} \in \mathbf{R}^{n^m}, \tilde{\boldsymbol{\tau}} = (\tilde{\tau}_1, \dots, \tilde{\tau}_{n^m})^{\top} \in \mathbf{R}^{n^m} \\ &\min \quad W = \sum_{i=1}^{n^m} \rho_i l_i \tilde{a}_i \\ &\text{s.t.} \quad -\boldsymbol{\Psi}_j(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c^j, \bar{u}_c^j) - \sum_{i=1}^{n^m} \tau_i \boldsymbol{\Omega}_i(\tilde{\boldsymbol{a}}, \eta_i) \succcurlyeq \mathbf{O},\ j = 1, \dots, m, \\ &\qquad 0 < \tilde{a}_i^l < \tilde{a}_i,\ i = 1, \dots, n^m, \\ &\qquad \tau_i \ge 0,\ i = 1, \dots, n^m. \end{aligned} \tag{2.28}$$

where $\boldsymbol{\Psi}_j, \boldsymbol{R}_c^j$ and $\bar{u}_c^j$ are shape matrix of the admissible set, localization matrix and upper bound associated with $g_j \leq 0, j = 1, \dots, m$, respectively.



A: attainable set

B: admissible set

Figure. 3 Attainable set and admissible set

It is worth noting that (2.28) is a mathematical programming with finite number of constraints. The sufficient condition in the lemma of S-procedure guarantees that the optimal solution (even a local optimum) will definitely satisfy the robust structural behavior constraint in (2.1). It can also be proved that the NLSDP problem in (2.28) will always have a feasible solution if $\tilde{a}_i,\ i = 1, \dots, n^m$ are sufficiently large. Therefore, confidence robust solution of WCDO can be obtained with use of this single-level problem formulation.

If the initial manufacture error is taken into consideration, the corresponding confidence single-level formulation can also be constructed in the following form

$$
\begin{aligned}
&\text{Find} \quad \lambda \in \mathrm{R}, \tilde{\boldsymbol{a}} = (\tilde{a}_1, \cdots, \tilde{a}_{n^m})^\top \in \mathbf{R}^{n^m}, \boldsymbol{\tau}^1 = \left(\tau_1^1, \dots, \tau_{n^m}^1\right)^\top \in \mathbf{R}^{n^m}, \boldsymbol{\tau}^2 = \left(\tau_1^2, \dots, \tau_{n^m}^2\right)^\top \in \mathbf{R}^{n^m} \\
&\text{Min} \quad \sum\nolimits_{i=1}^{n^m} \rho_i l_i \tilde{a}_i \\
&\text{s.t.} \quad -\boldsymbol{\Psi} \succcurlyeq \sum_{i=1}^{n^m} \tau_i^1 \boldsymbol{\Omega}_i^1(\tilde{\boldsymbol{a}}) + \sum_{i=1}^{n^m} \tau_i^2 \boldsymbol{\Omega}_i^2(\tilde{\boldsymbol{a}}) + \lambda \boldsymbol{\Omega}_f \\
&\qquad \tilde{a}_i \geq \underline{a}_i > 0,\ \tau_i^1 \geq 0,\ \tau_i^2 \geq 0,\ i = 1, \cdots, n^m
\end{aligned}
\tag{2.29}
$$

where

$$
\boldsymbol{\Omega}_i^1 = \begin{pmatrix} \mathbf{O} & & \\ & -(\tilde{a}_i \epsilon_i E_i)^{-2} \boldsymbol{D}_i & \\ & & \mathbf{O} & \\ & \text{sym.} & & (1-\eta_i)^2 \end{pmatrix}, \tag{2.30}
$$

$$
\boldsymbol{\Omega}_i^2(\tilde{\boldsymbol{a}}) = \begin{pmatrix} -\boldsymbol{D}_i & & & \\ & \mathbf{O} & & \\ & & \left(\frac{\tilde{a}_i \eta_i E_i}{l_i}\right)^2 \boldsymbol{r}_i \boldsymbol{r}_i^\top & \\ & \text{sym.} & & 0 \end{pmatrix}, \tag{2.31}
$$

$$\boldsymbol{\Omega}_f(\widetilde{\boldsymbol{a}}) = \begin{pmatrix} \boldsymbol{V}^\top\boldsymbol{V} & \boldsymbol{V}^\top\boldsymbol{V} & \boldsymbol{V}^\top\boldsymbol{K}(\widetilde{\boldsymbol{a}}) & -\boldsymbol{V}^\top\boldsymbol{f} \\ & \boldsymbol{V}^\top\boldsymbol{V} & \boldsymbol{V}^\top\boldsymbol{K}(\widetilde{\boldsymbol{a}}) & -\boldsymbol{V}^\top\boldsymbol{f} \\ & & \boldsymbol{K}(\widetilde{\boldsymbol{a}})\boldsymbol{K}(\widetilde{\boldsymbol{a}}) & -\boldsymbol{K}(\widetilde{\boldsymbol{a}})\boldsymbol{f} \\ & \text{sym.} & & \boldsymbol{f}^\top\boldsymbol{f} \end{pmatrix}, \tag{2.32}$$

and

$$\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{0} & & & \\ & \mathbf{0} & & \\ & & \boldsymbol{R}_c^\top\boldsymbol{R}_c & \\ & & & -\bar{u}_c \end{pmatrix}. \tag{2.33}$$

In Eq. (2.30) $\epsilon_i, i = 1, \dots, n^m$ are the maximum strain that induced by the manufacture error in i-th bar, respectively. For the limitation of space, the details of derivation are omitted here.

3. Confidence single-level formulation of robust design under steady state condition

In this subsection, confidence single-level formulation of robust design under steady state condition will be presented. As in the static case, only interval type stiffness uncertainties are considered. Robust optimal design is performed under the condition that the norms of the concerned structural responses (e.g. the displacement of a specified node) at steady state should not exceed the prescribed values. From the above analysis, it is known that the key point to construct the confidence single-level formulation is to express the admissible set in form of quadratic inequalities. This issue will be addressed in the following subsection. Unless otherwise noted, in the following derivations, all reappeared symbols will represent the same quantities as those in the previous section.

3.1 ATTAINABLE AND ADMISSIBLE SETS OF CONCERNED STRUCTURAL RESPONSE UNDER PRESCRIBED INTERVAL TYPE STIFFNESS UNCERTAINTIES

Let $\boldsymbol{f}e^{\mathrm{i}\omega t} \in \mathbf{C}^{n^d}$ denote the harmonic exciting load vector, the dynamic structural response can be described as

$$\boldsymbol{M}\ddot{\hat{\boldsymbol{u}}}(\boldsymbol{a}) + \boldsymbol{C}(\boldsymbol{a})\dot{\hat{\boldsymbol{u}}}(\boldsymbol{a}) + \boldsymbol{K}(\boldsymbol{a})\hat{\boldsymbol{u}}(\boldsymbol{a}) = \boldsymbol{f}e^{\mathrm{i}\omega t}, \tag{3.1}$$

where $\boldsymbol{K}(\boldsymbol{a}) \in \boldsymbol{S}^{n^d}$, $\boldsymbol{M} \in \boldsymbol{S}^{n^d}$ and $\boldsymbol{C}(\boldsymbol{a}) \in \boldsymbol{S}^{n^d}$ are the stiffness, mass (independent of $\boldsymbol{a}$) and damping matrices, respectively.

Let $\hat{\boldsymbol{u}}(\boldsymbol{a}) = \boldsymbol{u}(\boldsymbol{a})e^{\mathrm{i}\omega t}$, the steady state structural response can be solved with the following equation

$$\left(-\omega^2\boldsymbol{M} + \mathrm{i}\omega\boldsymbol{C}(\boldsymbol{a}) + \boldsymbol{K}(\boldsymbol{a})\right)\boldsymbol{u}(\boldsymbol{a}) = \boldsymbol{f}, \tag{3.2}$$

where $\boldsymbol{u}(\boldsymbol{a}) \in \mathbf{C}^{n^d}$. Furthermore, we have

$$\begin{pmatrix} -\omega^2\boldsymbol{M} + \boldsymbol{K}(\boldsymbol{a}) & -\omega\boldsymbol{C}(\boldsymbol{a}) \\ \omega\boldsymbol{C}(\boldsymbol{a}) & -\omega^2\boldsymbol{M} + \boldsymbol{K}(\boldsymbol{a}) \end{pmatrix}\begin{pmatrix} \mathbf{Re}\ \boldsymbol{u}(\boldsymbol{a}) \\ \mathbf{Im}\ \boldsymbol{u}(\boldsymbol{a}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \mathbf{0} \end{pmatrix}. \tag{3.3}$$

Denoting $\boldsymbol{v}(\boldsymbol{a}) \in \mathbf{R}^{2n^d}$ as

$$\boldsymbol{v}(\boldsymbol{a}) = \begin{pmatrix} \boldsymbol{v}_1(\boldsymbol{a}) \\ \boldsymbol{v}_2(\boldsymbol{a}) \end{pmatrix} = \begin{pmatrix} \mathbf{Re}\ \boldsymbol{u}(\boldsymbol{a}) \\ \mathbf{Im}\ \boldsymbol{u}(\boldsymbol{a}) \end{pmatrix} \tag{3.4}$$

and substituting Eq. (3.4) into Eq. (3.3), it yields that

$$\begin{pmatrix} \boldsymbol{K}(\boldsymbol{a}) & \\ & \boldsymbol{K}(\boldsymbol{a}) \end{pmatrix}\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} + \begin{pmatrix} -\omega^2\boldsymbol{M}\boldsymbol{v}_1 - \omega\boldsymbol{C}(\boldsymbol{a})\boldsymbol{v}_2 \\ \omega\boldsymbol{C}(\boldsymbol{a})\boldsymbol{v}_1 - \omega^2\boldsymbol{M}\boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \mathbf{0} \end{pmatrix}. \tag{3.5}$$

Let

$$\omega^2\boldsymbol{M}\boldsymbol{v}_1 + \omega\boldsymbol{C}(\boldsymbol{a})\boldsymbol{v}_2 = -\boldsymbol{w}_1, \tag{3.6}$$

$$\omega \boldsymbol{C}(\boldsymbol{a})\boldsymbol{v}_1 - \omega^2 \boldsymbol{M}\boldsymbol{v}_2 = \boldsymbol{w}_2 \tag{3.7}$$

and

$$\boldsymbol{p}_1(\Delta\boldsymbol{a}) = (\boldsymbol{I} + \boldsymbol{H}(\Delta\boldsymbol{a})\boldsymbol{V}^\top \boldsymbol{K}(\widetilde{\boldsymbol{a}})^{-1}\boldsymbol{V})^{-1}\boldsymbol{H}(\Delta\boldsymbol{a})\boldsymbol{V}^\top(\boldsymbol{K}(\widetilde{\boldsymbol{a}})^{-1}(\boldsymbol{f} - \boldsymbol{w}_1)), \tag{3.8}$$

$$\boldsymbol{p}_2(\Delta\boldsymbol{a}) = (\boldsymbol{I} + \boldsymbol{H}(\Delta\boldsymbol{a})\boldsymbol{V}^\top \boldsymbol{K}(\widetilde{\boldsymbol{a}})^{-1}\boldsymbol{V})^{-1}\boldsymbol{H}(\Delta\boldsymbol{a})\boldsymbol{V}^\top(\boldsymbol{K}(\widetilde{\boldsymbol{a}})^{-1}(-\boldsymbol{w}_2)), \tag{3.9}$$

Assuming that the damping matrix is $\boldsymbol{C}(\boldsymbol{a}) = 2\beta\boldsymbol{K}(\boldsymbol{a})$ (here $0 < \beta < 1$ is a constant), then with use of the same techniques developed in the previous subsection, the attainable set of the structural response can be expressed in terms of $\boldsymbol{q} = (\boldsymbol{p}_1^\top, \boldsymbol{w}_1^\top, \boldsymbol{p}_2^\top, \boldsymbol{w}_2^\top)^\top$ as follows

$$\mathfrak{D}_p(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \left\{ \boldsymbol{q} \in \mathbf{R}^{2(n^m + n^d)} \;\middle|\; \begin{matrix} \boldsymbol{\xi}^\top \boldsymbol{\Omega}_i^1(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta})\boldsymbol{\xi} \geq 0, \boldsymbol{\xi}^\top \boldsymbol{\Omega}_i^2(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta})\boldsymbol{\xi} \geq 0, \forall i = 1, \dots, n^m, \\ \boldsymbol{\xi}^\top \boldsymbol{\Omega}^3(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta})\boldsymbol{\xi} \geq 0, \boldsymbol{\xi}^\top \boldsymbol{\Omega}^4(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta})\boldsymbol{\xi} \geq 0 \end{matrix} \right\}, \tag{3.10}$$

where $\boldsymbol{\xi} = (\boldsymbol{q}^\top, 1)^\top$ and

$$\boldsymbol{\Omega}_i^1(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \begin{pmatrix} \alpha_i \boldsymbol{b}_i \boldsymbol{b}_i^\top - \boldsymbol{D}_i & \alpha_i \boldsymbol{b}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1} & & & -\alpha_i \boldsymbol{b}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & \alpha_i \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{r}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1} & & & -\alpha_i \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{r}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & & \mathbf{O} & & \\ \text{sym.} & & & \mathbf{O} & \\ & & & & \alpha_i \boldsymbol{f}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{r}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \end{pmatrix}, \tag{3.11}$$

$$\boldsymbol{\Omega}_i^2(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \begin{pmatrix} \mathbf{O} & & & & \\ & \mathbf{O} & & & \\ & & \alpha_i \boldsymbol{b}_i \boldsymbol{b}_i^\top - \boldsymbol{D}_i & \alpha_i \boldsymbol{b}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1} & \\ & \text{sym.} & & \alpha_i \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{r}_i \boldsymbol{r}_i^\top \widetilde{\boldsymbol{K}}^{-1} & \\ & & & & 0 \end{pmatrix}, \tag{3.12}$$

$$\boldsymbol{\Omega}^3(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \begin{pmatrix} \gamma^2 \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{V} & \gamma^2 \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1} - \theta\gamma \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M} & & \gamma \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M} & -\gamma^2 \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & \theta^2 \boldsymbol{I} - 2\theta\gamma \boldsymbol{M}\widetilde{\boldsymbol{K}}^{-1} + \gamma^2 \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1} & & \gamma \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M} - \theta \boldsymbol{I} & \theta\gamma \boldsymbol{M}\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} - \gamma^2 \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & & \mathbf{O} & & \\ & \text{sym.} & & \boldsymbol{I} & -\gamma \boldsymbol{M}\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & & & & \gamma^2 \boldsymbol{f}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \end{pmatrix}, \tag{3.13}$$

$$\boldsymbol{\Omega}^4(\widetilde{\boldsymbol{a}}, \boldsymbol{\eta}) = \begin{pmatrix} \mathbf{O} & & & & \\ & \boldsymbol{I} & -\gamma \boldsymbol{M}\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{V} & \theta \boldsymbol{I} - \gamma \boldsymbol{M}\widetilde{\boldsymbol{K}}^{-1} & -\boldsymbol{f} \\ & & \gamma^2 \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1}\boldsymbol{V} & \gamma^2 \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1} - \theta\gamma \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M} & \gamma \boldsymbol{V}^\top \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}\boldsymbol{f} \\ & \text{sym.} & & \theta^2 \boldsymbol{I} - 2\theta\gamma \boldsymbol{M}\widetilde{\boldsymbol{K}}^{-1} + \gamma^2 \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}^2\widetilde{\boldsymbol{K}}^{-1} & \gamma \widetilde{\boldsymbol{K}}^{-1}\boldsymbol{M}\boldsymbol{f} - \theta \boldsymbol{f} \\ & & & & \boldsymbol{f}^\top \boldsymbol{f} \end{pmatrix}, \tag{3.14}$$

where $\theta = 1/2\beta\omega$ and $\gamma = \omega^2\alpha$. In Eq. (3.13) and Eq. (3.14), $\boldsymbol{I}$ denotes the identity matrix with appropriate dimensions.

If the steady state nodal displacement is $\|u_c\|^2 \leq \overline{u}_c$, then the corresponding admissible sets can be constructed in terms of $\boldsymbol{q}$ as

$$\mathfrak{P} = \mathfrak{P}(\tilde{\boldsymbol{a}}; \boldsymbol{R}_c, \bar{u}_c) = \left\{ \boldsymbol{q} \in \mathbf{R}^{2(n^m+n^d)} \middle| -\boldsymbol{\xi}^\top \boldsymbol{\Omega}(\tilde{\boldsymbol{a}})\boldsymbol{\xi} \geq 0 \right\}, \tag{3.15}$$

where

$$\boldsymbol{\Omega}(\tilde{\boldsymbol{a}}) = \begin{pmatrix} \boldsymbol{V}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1}\boldsymbol{V} & \boldsymbol{V}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1} & & & -\boldsymbol{V}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & \tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1} & & & -\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1}\boldsymbol{f} \\ & & \boldsymbol{V}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1}\boldsymbol{V} & \boldsymbol{V}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1} & \\ & \text{sym.} & & \tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1} & \\ & & & & \boldsymbol{f}^\top\tilde{\boldsymbol{K}}^{-1}\boldsymbol{R}_c^\top\boldsymbol{R}_c\tilde{\boldsymbol{K}}^{-1}\boldsymbol{f} - \bar{u}_c \end{pmatrix}. \tag{3.16}$$

3.2 CONFIDENCE SINGLE-LEVEL FORMULATION OF ROBUST DESIGN UNDER STEADY STATE CONDITION

Based on the results derived in the previous subsection, the following confidence single-level problem formulation for steady state dynamic case can be obtained

$$\begin{aligned} &\text{find} \quad \tilde{\boldsymbol{a}} = (\tilde{a}_1, \cdots, \tilde{a}_{n^m})^\top, \boldsymbol{\tau}^1 = (\tau_1^1, \dots, \tau_{n^m}^1)^\top, \boldsymbol{\tau}^2 = (\tau_1^2, \dots, \tau_{n^m}^2)^\top, \lambda_1, \lambda_2 \\ &\text{Min} \quad \sum_{i=1}^{n^m} \rho_i l_i \tilde{a}_i \\ &\text{s.t.} \quad -\boldsymbol{\Omega}(\tilde{\boldsymbol{a}}) \succcurlyeq \sum_{i=1}^{n^m} \tau_i^1 \boldsymbol{\Omega}_i^1(\tilde{\boldsymbol{a}}) + \sum_{i=1}^{n^m} \tau_i^2 \boldsymbol{\Omega}_i^2(\tilde{\boldsymbol{a}}) + \lambda_1 \boldsymbol{\Omega}^3(\tilde{\boldsymbol{a}}) + \lambda_2 \boldsymbol{\Omega}^4(\tilde{\boldsymbol{a}}), \\ &\qquad \tilde{a}_i \geq \underline{a}_i > 0, \tau_i^1 \geq 0, \tau_i^2 \geq 0, \ i = 1, \cdots, n^m \end{aligned} \tag{3.17}$$

by applying the two lemmas mentioned above. Optimization problem in (3.17) is also a NLSDP problem, whose solution aspect will be discussed in the next section.

3.3 DEFICIENCY OF THE NAÏVE SINGLE-LEVEL FORMULATION FOR ROBUST OPTIMIZATION PROBLEMS

In the literatures, some authors proposed to replace the lower-level worst-case analysis optimization problem by the corresponding first order necessary K-K-T optimality conditions of it. In this way, the so-called single-level formulation can also be constructed. From the author's point of view, cautions should be made when this approach is applied to WCDO problems in which the lower-level program is non-convex. This is because K-K-T conditions are only necessary conditions to identify an optimal solution. A feasible design point that satisfies these conditions may not be a global optimum!

To explain this point more clearly, let us considered the first example discussed in the beginning of Section 2 again. For this problem, the naïve single-level formulation is as follows

$$\begin{aligned} &\text{find} \quad \tilde{a}_1 \in \mathrm{R}, \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_4)^\top \in \mathbf{R}^4, \ \boldsymbol{p} = (p_1, p_2)^\top \in \mathbf{R}^2 \\ &\text{min} \quad W = 2\tilde{a}_1 + \frac{1}{2} \\ &\text{s.t.} \quad u_1(p_1, 1) - u_1(\tilde{a}_1, 1) \leq 0.01694, \\ &\qquad u_1(\tilde{a}_1, 1) - u_1(p_2, 1) \leq 0.01694, \end{aligned} \tag{3.18}$$

$$\frac{\partial L_1}{\partial p_1} = \frac{\partial\left(-u_1(p_1,1) + \lambda_1(p_1 - 1.5\tilde{a}_1) + \lambda_2(0.5\tilde{a}_1 - p_1)\right)}{\partial p_1} = 0,$$
$$\lambda_1(p_1 - 1.5\tilde{a}_1) = 0, \lambda_1 \geq 0, p_1 - 1.5\tilde{a}_1 \leq 0,$$
$$\lambda_2(0.5\tilde{a}_1 - p_1) = 0, \lambda_2 \geq 0, 0.5\tilde{a}_1 - p_1 \leq 0,$$
$$\frac{\partial L_1}{\partial p_2} = \frac{\partial\left(u_1(p_2,1) + \lambda_3(p_2 - 1.5\tilde{a}_1) + \lambda_4(0.5\tilde{a}_1 - p_2)\right)}{\partial p_2} = 0,$$
$$\lambda_3(p_2 - 1.5\tilde{a}_1) = 0, \lambda_3 \geq 0, p_2 - 1.5\tilde{a}_1 \leq 0,$$
$$\lambda_4(0.5\tilde{a}_1 - p_2) = 0, \lambda_4 \geq 0, 0.5\tilde{a}_1 - p_2 \leq 0,$$
$$\tilde{a}_1 \geq 2.$$

It can be verified that $\tilde{a}_1 = 2, p_1 = 1.8059, p_2 = 3, \lambda_1 = \lambda_2 = \lambda_4 = 0, \lambda_3 = 0.0193$ is a global optimum of this *naive* single-level formulation. But $\tilde{a}_1 = 2$, is even not a feasible point of the original robust optimization problem as has been shown previously.

4. Solution of the NLSDP problem

The confidence single-level formulations obtained in the previous section are NLSDP problems. Compared with linear semi-definite programming problems, the solution of NLSDP problems has not yet arrived at its mature stage. In the present paper, a modified Augmented Lagrange multiplier method is proposed to solve the corresponding NLSDP problems. This is an improved version of the algorithm proposed in [Sun, Sun and Zhang (2008)]. In the following, this method will be described briefly.

The general NLSDP problem we want to solve is

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & h_i(\boldsymbol{x}) \geq 0, \quad i = 1, \cdots, m, \\ & \boldsymbol{g}_j(\boldsymbol{x}) \succcurlyeq \boldsymbol{0}, \quad j = 1, \cdots, n, \end{aligned} \tag{4.1}$$

where f and h_i , $\boldsymbol{g}_j, i = 1, \ldots, m,\ j = 1, \ldots, n$ are all twice continuously differentiable.

The following notation and terminology are used throughout the following discussions. For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbf{R}^{n\times n}$, we use the inner product $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{Tr}(\boldsymbol{A}^\top \boldsymbol{B})$ and the Frobenius norm $\|\cdot\|_F$ induced by the inner product, where the symbol Tr stands for the trace of $\boldsymbol{A}^\top \boldsymbol{B}$. $\Pi_{\mathbf{S}_+^p}(\cdot)$ is the metric projection operator onto the set $\mathbf{S}_+^p$. By constructing the following

$$L_c(\boldsymbol{x}, \boldsymbol{U}, c) = f(\boldsymbol{x}) + \frac{1}{2c}\sum_{j=1}^{n}\left[\left\|\prod\nolimits_{\mathbf{S}_+^p}\left(\boldsymbol{U}_j - c\boldsymbol{g}_j(\boldsymbol{x})\right)\right\|_F^2 - \left\|\boldsymbol{U}_j\right\|_F^2\right], \left(\boldsymbol{x}, \boldsymbol{U}_j, c\right) \in \mathbf{R}^n \times \mathbf{S}^p \times \mathrm{R}^+$$

augmented Lagrangian function, the flow-chart of the modified augmented Lagrangian algorithm can be listed as follows

Step 0. Let $c^0 > 0$, $\bar{c} > c^0$ and $\kappa > 1$ are all real numbers, N_0 is a positive integer, $\boldsymbol{U}_j^0 \in \mathbf{S}_+^p$, $j = 1, \ldots, n$ and set $k := 0$.

Step 1. $\boldsymbol{x}^k$ is obtained by

$$\boldsymbol{x}^k = \operatorname{argmin} L_c\left(\boldsymbol{x}, \boldsymbol{U}_j, c\right) \quad \text{s.t. } h_i(\boldsymbol{x}) \geq 0,\ i = 1, \cdots, m. \tag{4.2}$$

Step 2. If $\mathrm{Tr}\left(\boldsymbol{U}_j \boldsymbol{g}_j\left(\boldsymbol{x}^k\right)\right) = 0$ and $\lambda_{\min}\left(\boldsymbol{g}_j\left(\boldsymbol{x}^k\right)\right) \geq 0$, then stop and $\boldsymbol{x}^k$ is a stationary point of the NLSDP problem in (4.1). Otherwise, goto **Step 3**.

Step 3. Update $\boldsymbol{U}_j$ by

$$\boldsymbol{U}_j^{k+1} = \prod_{\mathbf{S}_+^p} \left(\boldsymbol{U}_j^k - c^k \boldsymbol{g}_j(\boldsymbol{x}^k)\right), \ j = 1, \dots, n.$$

Update c by

$$c^{k+1} = \begin{cases} \kappa c^k, & k < N_0 \\ \bar{c}, & \text{otherwise} \end{cases}.$$

Step 4. Set $k := k + 1$ and goto **Step 1**.

The above modified Augmented Lagrange multiplier method has sound convergence properties. Compared with the algorithm developed in [Sun, Sun and Zhang (2008)] where nonlinear constraints are treated by putting them into the diagonal of the matrix inequality, in our approach a constrained optimization is solved for finding $\boldsymbol{x}^k$. This greatly reduces the size of the matrix inequality constraint which makes the present approach very suitable to solve NLSDP problems with a large number of nonlinear constraints. Numerical examples presented in the next section show that the solutions of the NLSDP problems very efficiently with the proposed algorithm.

5. Numerical examples

In this section, two numerical examples will be presented to illustrate the numerical performance of the proposed approach for confidence robust optimization of truss structures under static and steady state conditions, respectively.

5.1 ROBUST OPTIMIZATION UNDER STATIC CONDITION

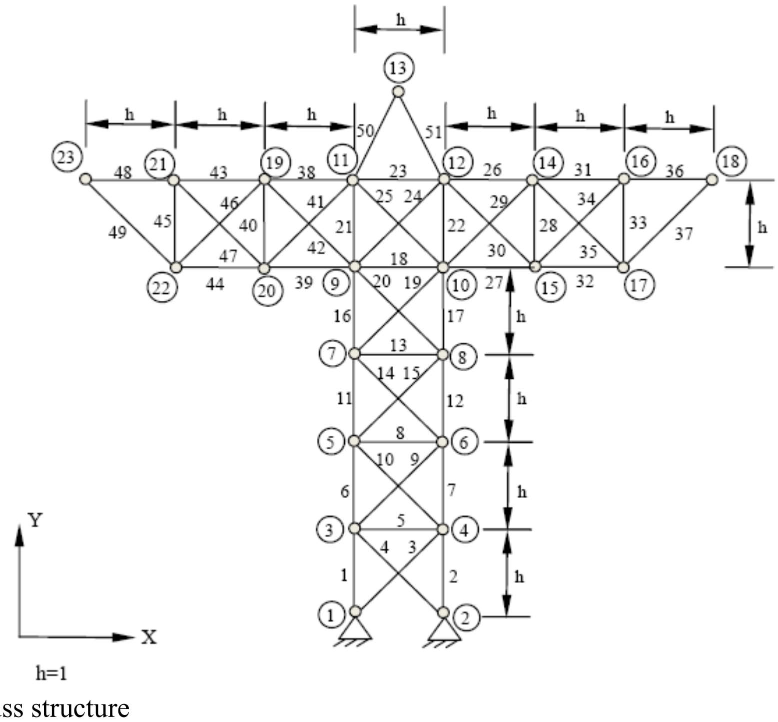


Figure.4 A 51-bar truss structure

In this example, a 51-bar truss structure shown in Fig. 4 is examined. The Young's moduli and the densities of the bars are $E_i = 1.0,\ i = 1,\cdots,51$ and $\rho_i = 1.0,\ i = 1,\cdots,51$, respectively. Four identical external loads $\boldsymbol{f} = (0.1,\ 0.0)^{\mathrm{T}}$ are applied at nodes 3,5,7,9 and 11, respectively. The objective is to minimize the weight of the structure under stiffness uncertainty ($(1-\eta_i)\tilde{a}_i \le a_i \le (1+\eta_i)\tilde{a}_i,\ i = 1,2,\dots,51$). The structural performance constraint is $u_{13x}^2 \le 0.49$. In this and the next example, the design variables are $\tilde{a}_i's$ (the nominal values of the cross sectional areas of bars). The lower bounds on $\tilde{a}_i's$ are all taken as 0.1.

The optimal cross sectional areas of each bar for $\eta_i = 0.5,\ i = 1,\dots,51$ are listed in Table. I. Fig.6 plots the iteration history of this problem. In this and the next examples, the convergence criteria adopt are such that the minimum eigenvalue is less than 10^{-6}. Note that if the initial design is infeasible for the optimization problem in (4.2), then our algorithm will find a feasible solution automatically. So the points depicted in the iteration history figures are all corresponding to the feasible solutions of (4.2).

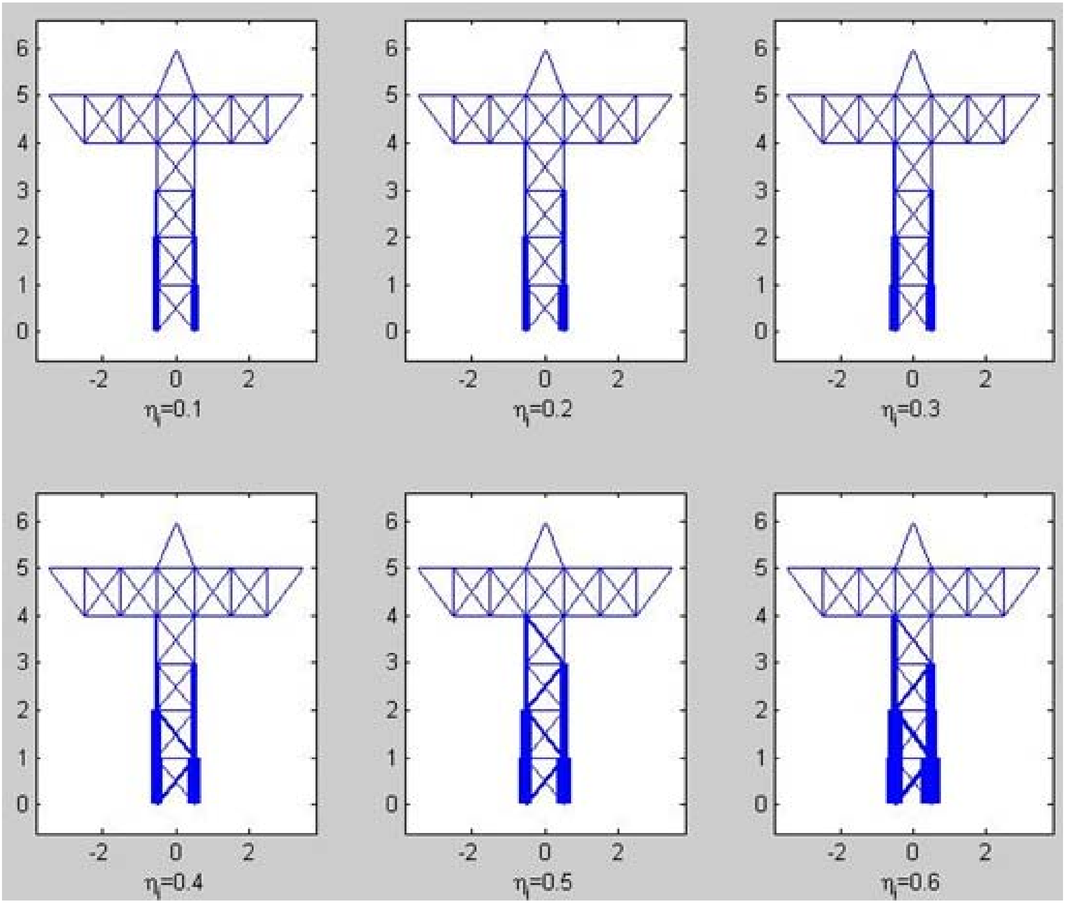


Figure.5 Optimal designs for 51-bar truss structure

Table.I Optimal solution for $\eta_i = 0.6$

Area		Area		Area	
Weight	1245.1535	$\tilde{a}_1$	159.6457	$\tilde{a}_2$	186.4051
$\tilde{a}_3$	52.7866	$\tilde{a}_4$	10.3331	$\tilde{a}_5$	0.1000
$\tilde{a}_6$	141.9707	$\tilde{a}_7$	108.2469	$\tilde{a}_8$	0.1000
$\tilde{a}_9$	6.0988	$\tilde{a}_{10}$	52.3919	$\tilde{a}_{11}$	70.5297
$\tilde{a}_{12}$	95.0666	$\tilde{a}_{13}$	0.1000	$\tilde{a}_{14}$	5.8513
$\tilde{a}_{15}$	43.1570	$\tilde{a}_{16}$	62.8346	$\tilde{a}_{17}$	29.0020
$\tilde{a}_{18}$	15.8022	$\tilde{a}_{19}$	0.1000	$\tilde{a}_{20}$	41.7276
$\tilde{a}_{21}$	25.5592	$\tilde{a}_{22}$	6.3910	$\tilde{a}_{23}$	0.1000
$\tilde{a}_{24}$	4.7710	$\tilde{a}_{25}$	22.2403	$\tilde{a}_{26}$	0.1000
$\tilde{a}_{27}$	0.1000	$\tilde{a}_{28}$	0.1000	$\tilde{a}_{29}$	0.1000
$\tilde{a}_{30}$	0.1000	$\tilde{a}_{31}$	0.1000	$\tilde{a}_{32}$	0.1000
$\tilde{a}_{33}$	0.1000	$\tilde{a}_{34}$	0.1000	$\tilde{a}_{35}$	0.1000
$\tilde{a}_{36}$	0.1000	$\tilde{a}_{37}$	0.1000	$\tilde{a}_{38}$	0.1000
$\tilde{a}_{39}$	0.1000	$\tilde{a}_{40}$	0.1000	$\tilde{a}_{41}$	0.1000
$\tilde{a}_{42}$	0.1000	$\tilde{a}_{43}$	0.1000	$\tilde{a}_{44}$	0.1000
$\tilde{a}_{45}$	0.1000	$\tilde{a}_{46}$	0.1000	$\tilde{a}_{47}$	0.1000
$\tilde{a}_{48}$	0.1000	$\tilde{a}_{49}$	0.1000	$\tilde{a}_{50}$	0.8235
$\tilde{a}_{51}$	0.8235	Iter.	**15**		

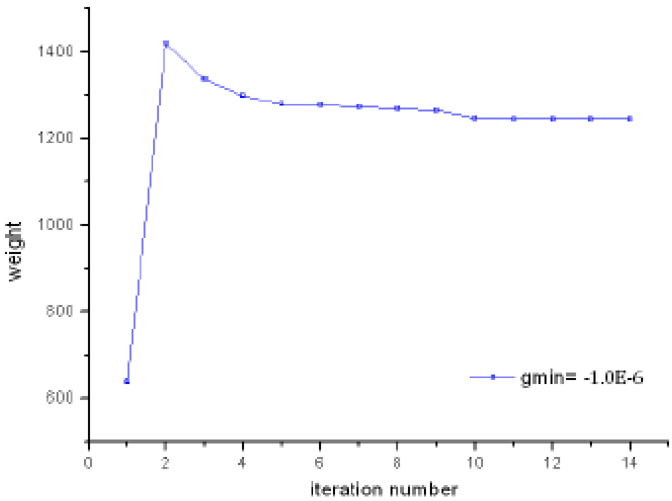


Figure.6 Iteration history for 51-bar truss example

Furthermore, global optimization method developed in [see Guo, Bai and Zhang (2008)] is also employed to examine the feasibilities of the obtained solutions. Analysis results show that for all of the optimal solutions obtained by the proposed single-level NLSDP formulation, the considered worst case behavior constraints all become active. This indicates that for this problem, in fact no gap has been introduced by the S-procedure operation in this example.

5.2 ROBUST OPTIMIZATION UNDER STEADY STATE CONDITION

In this example, a 29-bar truss structure shown in Fig. 7 is examined. The Young's moduli and the densities of the bars are $E_i = 10000,\ i = 1,\cdots,29$ and $\rho_i = 0.00786,\ i = 1,\cdots,29,$ respectively. A harmonic external load $\boldsymbol{f} = (10.0,\ 10.0)^{\top} e^{\mathrm{i}0.4t}$ is applied at node 11. The mass matrix is taken as $\boldsymbol{M} = 10.0\boldsymbol{I}_{20\times 20}$ and $\beta = 0.2$. The objective is to minimize the weight of the structure under stiffness uncertainty ($(1-\eta_i)\tilde{a}_i \le a_i \le (1+\eta_i)\tilde{a}_i,\ i = 1,2,\dots,51$). The considered structural performance constraint is $u_{12y}^2 \le 0.1$ at steady state.

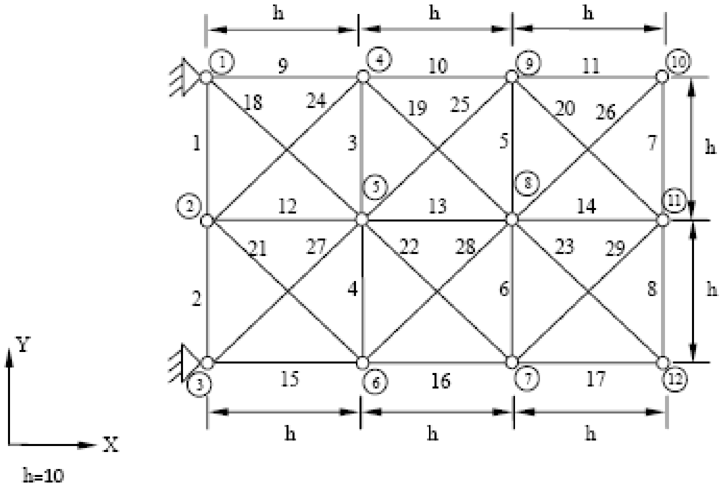


Figure.7 A 29-bar truss structure

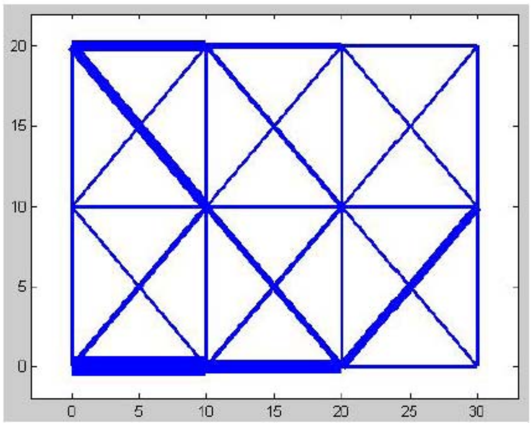


Figure.8 Optimal design for 29-bar truss structure

Table. II Optimal solution for $\eta_i = 0.1$					
Area		Area		Area	
Weight	0.4147	$\tilde{a}_1$	0.1000	$\tilde{a}_2$	0.1000
$\tilde{a}_3$	0.1000	$\tilde{a}_4$	0.1000	$\tilde{a}_5$	0.1000
$\tilde{a}_6$	0.1000	$\tilde{a}_7$	0.1000	$\tilde{a}_8$	0.1000
$\tilde{a}_9$	0.3014	$\tilde{a}_{10}$	0.1689	$\tilde{a}_{11}$	0.1000
$\tilde{a}_{12}$	0.1000	$\tilde{a}_{13}$	0.1000	$\tilde{a}_{14}$	0.1000
$\tilde{a}_{15}$	0.5573	$\tilde{a}_{16}$	0.4160	$\tilde{a}_{17}$	0.1000
$\tilde{a}_{18}$	0.2959	$\tilde{a}_{19}$	0.1272	$\tilde{a}_{20}$	0.1000
$\tilde{a}_{21}$	0.1000	$\tilde{a}_{22}$	0.2089	$\tilde{a}_{23}$	0.1097
$\tilde{a}_{24}$	0.1000	$\tilde{a}_{25}$	0.1000	$\tilde{a}_{26}$	0.1000
$\tilde{a}_{27}$	0.1661	$\tilde{a}_{28}$	0.1382	$\tilde{a}_{29}$	0.2445
Iter.	**15**				

For $\eta_i = 0.1, i = 1, \ldots, 29$, the optimization problem is solved with use of the single-level NLSDP formulation in (3.17). with $\tilde{a}_i^0 = 1.0,\ i = 1, \cdots, 29$, $(\tau^1)_i^0 = 1.0,\ i = 1, \cdots, 29$, $(\tau^2)_i^0 = 1.0,\ i = 1, \cdots, 29$, $\lambda_1 = 1.0$ and $\lambda_2 = 1.0$, respectively. The corresponding optimal structure is depicted in Fig. 8. The optimal cross sectional areas of the bars are listed in Table. II. The iteration history is shown in Fig. 9.

Verification analysis also shows that when

$$a_i = \begin{cases} \tilde{a}_i(1.0 + \eta_i), & \text{for } i = 5,6,8,12,13,14; \\ \tilde{a}_i(1.0 - \eta_i), & \text{otherwise.} \end{cases}$$

(where the values of $\tilde{a}_i, i = 1, \ldots, 29$ are listed in Table. II), the value of u_{12y} is $u_{12y} = 0.3099 - 0.0630\mathrm{i}$ and $u_{12y}^2 = 0.1$. As in the previous example, this means that no gap has been introduced by the S-procedure operation.

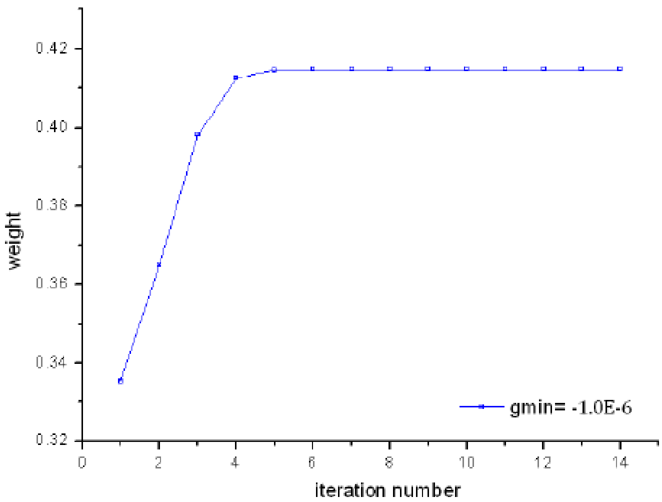


Figure.9 Iteration history for 29-bar truss example

6. Concluding remarks

In this paper, confidence single-level formulations for robust structural optimization under non-probabilistic stiffness uncertainties are presented. Both static and steady state conditions are considered. It is pointed out that finding the worst case structural performance is usually a non-convex optimization problem, then it is necessary to solve it with global optimality otherwise the feasibility of a given design cannot be evaluated correctly. This issue has not been well addressed in the literatures. Most of the algorithms used in the previous studies for obtaining the most unfavorable structural responses can only guarantee the convergence to a local optimum. Sometimes, this may underestimate the potential dangers of a given design severely. It is also suggested that cautions should be made when K-K-T optimality conditions are used to construct the single level formulation of WCDO problems since K-K-T conditions are only necessary conditions to identify a local optimal solution. Solutions that satisfy these conditions may not be the global optimum of non-convex optimization problems. The advantage of the proposed NLSDP single-level formulations consists in their reliability and efficiency for obtaining confidence robust optimal solutions. Although only truss structures are considered in the present paper, the proposed solution approaches can also be applied to robust optimal design of continuum structures discretized by finite elements. The current version of the proposed approaches is only applicable to linear elastic structures under small deformations. However, generalizing the methodologies proposed in this paper to non-linear case is a very interesting research topic, which deserves further explorations.

Acknowledgements

The financial supports from the National Natural Science Foundation (10772037, 10472022, 10332020 and 10925209), 973 Project of China (2006CB601205) and (2010CB832703) are gratefully acknowledged.

References

Royset J.O., Der Kiureghian A. and Polak E. Reliability-based optimal design of series structural systems. *Journal of Engineering Mechanics*, 127: 607-614, 2001.

Choi K.K., Tu J. and Park Y.H. Extensions of design potential concept for reliability-based design optimization to nonsmooth and extreme cases. *Structural and Multidisciplinary Optimization*, 22: 335-350, 2001.

Jung D.H. and Lee B.C. Development of a simple and efficient method for robust optimization. *International Journal for Numerical Methods in Engineering,* 53: 2201-2215, 2002.

Papadrakakis M. and Lagaros N.D. Reliability-based structural optimization using neural networks and Monte-Carlo simulation. *Computer Methods in Applied Mechanics and Engineering*, 191: 3491-3507, 2002.

Kharmanda G., Olhoff N., Mohamed A. and Lemaire M. Reliability-based topology optimization. *Structural and Multidisciplinary Optimization*,26: 295-307, 2004.

Lee K.H. and Park G.J. Robust optimization considering tolerances of design variables. *Computer and Structures*, 79: 77–86,2001.

Sandgren E. and Cameron T.M. Robust design optimization of structures through consideration of variation. *Computer and Structures*, 80: 1605–1613, 2002.

Lee K.H., Eom I.S., Park G.J. and Lee W.I. Robust design for unconstrained optimization problems using the Taguchi method. *AIAA Journal,* 34: 1059-1063, 1996.

Lee K.H. and Park G.J. Robust optimization in discrete design space for constrained problems. *AIAA Journal*, 40: 774-780,2002.
Elishakoff I.E. Essay on uncertainties in elastic and viscoelastic structures: from A.M. Freudental's criticisms to modern convex modeling. *Computers and Structures,* 56: 871-895, 1995.
Ben-Haim Y. and Elishakoff I.E. *Convex models of uncertainties in applied mechanics. Studies in Applied Mechanics,* No.25. Elsevier, New York, 1990.
Elishakoff I., Haftka R.T. and Fang J. Structural design under bounded uncertainty Optimization with anti-optimization, *Computers and Structures,* 53: 1401-1405, 1994.
Pantelides C.P. and Ganzerli S. Design of trusses under uncertain loads using convex models. *Journal of Structural Engineering,* 124: 318-329, 1998.
Lombardi M. and Haftka R.T. Anti-optimization techniques for structural design under load uncertainties. *Computer Methods in Applied Mechanics and Engineering,* 157: 19-31, 1998.
Au F.T.K., Cheng Y.S., Tham L.G. and Zeng G.W. Robust design of structures using convex models. *Computers and Structures,* 81: 2611-2619, 2003.
Gurav S.P., Goosen J.F.L. and VanKeulen F. Bounded-but-unknown uncertainty optimization using design sensitivities and parallel computing: application to MEMS. *Computers and Structures,* 83: 1134-1149, 2005.
Jiang C., Han X. and Liu G.R. Optimization of structures with uncertain constraints based on convex model and satisfaction degree of interval. *Computer Methods in Applied Mechanics and Engineering,* 196: 4791-4800, 2007.
Liang J.H. and Mourelatos P.J. System reliability-based design using a single-loop method. Reliability and Robust Design in Automotive Engineering, SAE TECHNICAL PAPER SERIES SP-2119, 2007.
McDonald M. and Mahadevan S. Reliability-based optimization with discrete and continuous decision and random variables. *Journal of Mechanical Design,* 130(6): 061-401, 2008.
Kuschel N. and Rackwitz R. Two basic problems in reliability-based structural optimization. *Mathematical Methods of Operations Research,* 46: 309-333, 1997.
Agarwal H. and Renaud J.E. A unilevel method for reliability based design optimization. 45th AIAA/ASME/ASCE/AHS/ASC Structures, *Structural Dynamics and Materials Palm Springs,* CA, 2004.
Kanno Y. and Takewaki I. Sequential semi-definite program for maximum robustness design of structures. *Journal of Optimization Theory and Applications,* 130: 265-287, 2006.
Boyd S. and Vandenberghe L. *Convex Optimization.* Cambridge University Press, Cambridge, 2004.
Sun D.F., Sun J. and Zhang L.W. The rate of convergence of the augmented Lagrangian method for nonlinear semi-definite programming. *Mathematical Programming,* 114: 349-391, 2008.
Guo X., Bai W. and Zhang W.S. Extreme structural response analysis of truss structures under material uncertainty via linear mixed 0-1 programming. *International Journal for Numerical Methods in Engineering*, 76: 253-277, 2008.