On The Stability And Controllability Of Fuzzy Control Set Differential Equations

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Abstract. Recently, the field of differential equations has been studying in a very abstract method. Instead of considering the behaviour of one solution of a differential equations, one studies its sheaf-solution and especially, studies fuzzy differential equations (a differential equations whose variables and derivative are fuzzy sets, see (Lakshmikantham and Mohapatra, 2003)-(Phu and Tung, 2008)).

In this paper, a fuzzy differential equations are generalized to be fuzzy set control differential equations (FSCDE) and we present the problem of stability and controllability of FSCDE. The paper is a continuation of our works in this direction (see (Phu, Quang and Tung, 2008)-(Phu and Dung, to appear)) for FSCDE.

Keywords: Fuzzy set theory; Set control differential equations; Fuzzy differential equations.

1. Introduction

In (Lakshmikantham and Mohapatra, 2003), the authors considered fuzzy differential equations (FDE) and had some important results on existence and comparison of solutions of FDE:

$$D_H u(t) = f(t, u(t)) \tag{1}$$

where $u(t_0) = u_0 \in H_0 \subset E^n, u(t) =\in E^n, t \in [t_0, T] = I \subset \mathbb{R}_+$ and $f : I \times E^n \to E^n$.

In this paper, we consider a fuzzy set control differential equations (FSCDE) as following:

$$D_{Ht} u(t) = f(t, u(t), v(t)) \tag{2}$$

where $u(t_0) = u_0 \in H_0 \subset E^{nN}, u(t) \in E^{nN}, v(t) \in E^{nP}, t \in [t_0, T] = I \subseteq \mathbb{R}_+$ and $f : I \times E^{nN} \times E^{nP} \to E^{nN}$. For this FSCDE we study the problems of stability and controllability.

The paper is organized as follows: in section 2, we recall some basic concepts and notations which are useful in next sections. In sections 3 we present the problem of stability and controllability of FSCDE in many difference cases.

2. Preliminaries

We recall some notations and concepts presented in detail in recent series works of Lakshmikantham V. et al... (See (Lakshmikantham and Mohapatra, 2003), (Lakshmikantham and Leela, 2001)).

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Let $K_C(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Given A, B in $K_C(\mathbb{R}^n)$, the Hausdorff distance between A and B defined as

$$D[A,B] = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{R^n}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{R^n}\}$$

where $\|.\|_{R^n}$ denotes the Euclidean norm in $\mathbb{R}^n$.

It is known that $(K_C(\mathbb{R}^n), D)$ is a complete metric space and if the space $K_C(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_C(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

The set $[\omega]^\alpha = \{x \in \mathbb{R}^n : \omega(x) \geq \alpha, 0 < \alpha \leq 1\}$ is called the α-level set.
Set $E^n = \{\omega : \mathbb{R}^n \to [0,1]$ such that $\omega(x)$ satisfies (i)-(iv) stated below$\}$

(i) ω is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $\omega(x_0) = 1$;

(ii) ω is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$\omega(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\omega(x_1), \omega(x_2)\};$$

(iii) ω is upper semicontinuous;

(iv) $[\omega]^0 = cl\{x \in \mathbb{R}^n : \omega(x) > 0\}$ is compact, then each it's element $\omega \in E^n$ is called a fuzzy number of fuzzy set.

Let us denote

$$d[\omega_1, \omega_2] = \sup\{d[[\omega_1]^\alpha, [\omega_2]^\alpha] : 0 \leq \alpha \leq 1\}$$

the distance between ω_1 and ω_2 in E^n, where $d[[\omega]^\alpha, [\omega]^\alpha]$ is Hausdorff distance between two set $[\omega_1]^\alpha, [\omega_2]^\alpha$ of $K_C(\mathbb{R}^n)$. Then (E^n, d) is a complete space. Some properties of metric d are as follows.

$$\begin{aligned} d[\omega_1+\omega_3, \omega_2+\omega_3] &= d[\omega_1, \omega_2], \\ d[\lambda\omega_1, \lambda\omega_2] &= |\lambda| d[\omega_1, \omega_2], \\ d[\omega_1, \omega_2] &\leq d[\omega_1, \omega_3] + d[\omega_3, \omega_2], \end{aligned}$$

for all $\omega_1, \omega_2, \omega_3 \in E^n$ and $\lambda \in \mathbb{R}$.

Given an interval $I = [t_0, T] \subseteq \mathbb{R}_+$. We give some definitions on the fuzzy mapping set: $u_i : I \to E^n, u_i(t) \in E^n$ where $[u_i(t)]^\alpha \in K_C(\mathbb{R}^n)$, and $u(t) = u_1(t) \times u_2(t) \times \ldots \times u_N(t) \in E^{nN} = E^n \times E^n \times \ldots \times E^n$, where every $u_i(t) \in E^n, i = 1, 2, \ldots, N$. The fuzzy set must be $u(t) = (u_1(t), u_2(t), \ldots, u_N(t))$.

Let $\bar{u}, u \in E^{nN}$. The set $z \in E^{nN}$ satisfying $\bar{u} = u + z$ is known as the geometric difference of the set $\bar{u}$ and u and is denoted by the symbol $\bar{u} - u$.
We have some possibilities to measure the new fuzzy variables $u, \bar{u}, f$ that are

$$d_0[u, \bar{u}] = \sum_{i=1}^{N} d[u_i, \bar{u}_i]$$

or

$$d_0[u, \bar{u}] = \frac{1}{N}\sqrt{\sum_{i=1}^{N} d^2[u_i, \bar{u}_i]}$$

or

$$d_0[u, \bar{u}] = (d[u_1, \bar{u}_1], d[u_2, \bar{u}_2], \ldots, d[u_N, \bar{u}_N])$$

and employ the metric space (E^{nN}, d_0) as a fuzzy Hausdorff metric space, where if $u \in E^{nN}$, then $\|u\| = d_0[u, \theta^{nN}]$

Anologously (Phu and Dung, to appear), we say that fuzzy mapping set $u(t) \in E^{nN}$ has a Hukuhara derivative $D_{Ht}u(t)$ at a point t, if

$$\lim_{\tau \to 0^+} \frac{u(t+\tau) - u(t)}{\tau} \quad \text{and} \quad \lim_{\tau \to 0^+} \frac{u(t) - u(t-\tau)}{\tau}$$

exist in the topology of E^{nN} and are equal to $D_{Ht}u(t)$. Here limits are taken in the metric space (E^{nN}, d_0):

$$\lim_{\tau \to 0^+} d_0\left[\frac{u(t+\tau) - u(t)}{\tau}, D_{Ht}u(t)\right] = 0$$

and

$$\lim_{\tau \to 0^+} d_0\left[\frac{u(t) - u(t-\tau)}{\tau}, D_{Ht}u(t)\right] = 0.$$

We say that $f : I \times E^{nN} \to E^{nN}$ has a Hukuhara derivative $D_{Hu}f(t, u(t))$ at a point set $(t, u(t))$, if exists fuzzy set $H \in E^{nN}$, $\|H\| = h$ such that

$$\lim_{h \to 0^+} \frac{f(t, u+H) - f(t, u(t))}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(t, u(t)) - f(t, u-H)}{h}$$

exist in the topology of E^{nN} and are equal to $D_{Hu}f(t, u(t))$.
Here limits are taken in the metric space .

$$\lim_{h \to 0^+} d_0\left[\frac{f(t, u+H) - f(t, u(t))}{h}, D_{Hu}f(t, u(t))\right] = 0$$

and

$$\lim_{h \to 0^+} d_0\left[\frac{f(t, u(t)) - f(t, u-H)}{h}, D_{Hu}f(t, u(t))\right] = 0.$$

More details in continuity, Hukuhara derivative, Hukuhara integral of the mapping $f : I \to E^{nN}$, please see (Phu and Dung, to appear).

3. Main Results

In fuzzy Hausdorff metric space E^{nN} we consider FSCDE, as following

$$D_{Ht}u(t) = f(t, u(t), v(t)), \quad u(t_0) = u_0 \in E^{nN} \tag{3}$$

where $f : I \times E^{nN} \times E^{nP} \to E^{nN}$ is fuzzy mapping set that is differentiable, fuzzy state $u(t) \in E^{nN}$, fuzzy control set $v(t) \in E^{nP}$.
The fuzzy mapping set $u \in C^1[I, E^{nN}]$ is said to be a solution of (3) on I if it satisfies (3) par on Hukuhara derivative D_{Ht} by t.

Definition 1. Set $v(t) \in E^{nP}$ will be an admissible fuzzy control set, that means at moment t_0 we have $u(t_0) = u(t_0, u_0, t, v(t_0)) = u_0 \in E^{nN}$, for any $\bar{u} \in E^{nN}$ exists $t_1 > t_0$ such that

$$u(t_1) = u(t_0, u_0, t_1, v(t_1)) = \bar{u}$$

and the pair of fuzzy set states $(u_0, \bar{u}) \in E^{nN}$ is called controllable by $v(t)$.

3.1. Problem Of Stability

We assume that FSDE (3) has the trivial set solution, that means $f(t, \theta^{nN}, v(t)) = \theta^{nN}$.
Put $S(r) = \{u(t) \in E^{nN} : d_0[u(t), \theta^{nN}] < r\}$ - neighbourhood of the zero set point.

Definition 2. The trivial set solution of FSCDE (3) is said to be

(LS) stable by Lyapunov's mean if for each $\varepsilon > 0$ and $t_0 > 0$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that $d_0[u_0, \theta^{nN}] < \delta$ implies $d_0[u(t), \theta^{nN}] < \varepsilon$, for $t \geq t_0$;

(ALS) asymptotically stable by Lyapunov's mean if it is stable and $\lim\limits_{t \to +\infty} d_0[u(t), \theta^{nN}] = 0$;

(ELS) exponentially stable by Lyapunov's mean if

$$d_0[u(t), \theta^{nN}] \leq \beta(d_0[u_0, \theta^{nN}], t_0) \exp[-\alpha(t - t_0)], t > t_0$$

where $\beta(d_0[.,.], t_0) : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$.

Theorem 1. Assume that the positive Lyapunov - like function $V \in C[\mathbb{R}_+ \times E^{nN} \times E^{nP}, \mathbb{R}_+]$ which satisfies the following conditions:

i) $|V(t, u(t), v(t)) - V(t, \bar{u}(t), \bar{v}(t))| \leq L(d_0[u(t), \bar{u}(t)] + d_0[v(t), \bar{v}(t)])$, where L is bounded Lipschitz constant, for all $u(t), \bar{u}(t) \in E^{nN}, v(t), \bar{v}(t) \in E^{nP}$ and $t \in \mathbb{R}_+$;

ii) $b(d_0[u(t), \theta^{nN}]) \leq V(t, u(t), v(t)) \leq a(t, d_0[u(t), \theta^{nN}])$, for $(t, u, v) \in \mathbb{R}_+ \times S^c(r) \times E^{nP}$ where $b(.), a(t, .)$ are increassing functions;

iii)

$$\begin{aligned} D^+V(t, u(t), v(t)) &\equiv \lim_{\tau \to 0^+} \sup \frac{1}{\tau} \Big\{ V(t + \tau, u(t) + \tau f(t, u(t), v(t)), v(t)) - V(t, u(t), v(t)) \Big\} \\ &\leq g(t, V(t, u(t), v(t))) \end{aligned}$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$, $g(t, 0) = 0$ for all $u(t) \in E^{nN}, v(t) \in E^{nP}$ and $t \in \mathbb{R}_+$.

a. If $g(t, V(t, u, v)) \leq 0, \forall t \geq t_0$ then a trivial set solution of FSCDE (3) is (LS).

b. If $g(t, V(t, u(t), v(t))) < 0, \forall t \geq t_0$ (or if $g(t, V(t, u(t), v(t))) < -\beta V, \forall t \geq t_0$) then a trivial set solution of FSCDE (3) is (ALS).

Proof. Setting the function $m(t) = V(t, u(t), v(t))$, we have

$$\begin{aligned} D^+m(t) &= D^+V(t, u(t), v(t)) \\ &\equiv \lim_{\tau \to 0^+} \sup \frac{1}{\tau}\Big\{V(t+\tau, u(t) + \tau f(t, u(t), v(t)), v(t)) - V(t, u(t), v(t))\Big\} \\ &\leq g(t, V(t, u(t), v(t))), \end{aligned}$$

so $D^+m(t) \leq g(t, m(t))$, implies that $m(t_0) \leq W_0$. Since $m(t) \leq r(t_0, W_0, t)$ where $r(t_0, W_0, t)$ is maximal solution of scarlar equation

$$\frac{dW}{dt} = g(t, W)$$

then $V(t, u(t), v(t)) \leq V(t_0, u_0, v_0)$.

Let $0 < \varepsilon < r,\ t_0 \in \mathbb{R}_+$, be given. Choose a $\delta = \delta(t_0, \varepsilon)$ such that $a(t_0, \delta) < b(\varepsilon)$. We claim that with this δ then (LS) holds.

If not, there is exists $u(t) = u(t_0, u_0, t, v(t))$ of FSCDE (3) and $t_1 > t_0$, such that $d_0[u(t_1), \theta^{nN}] = \varepsilon$ and $d_0[u(t), \theta^{nN}] \leq \varepsilon < r,\ t_0 < t < t_1$.
Wherenever $d_0[u_0, \theta^{nN}] < \delta$, because

$$V(t, u(t), v(t)) \leq V(t_0, u_0, v_0),\ t_0 < t < t_1$$

then

$$\begin{aligned} b(\varepsilon) = b(d_0[u(t_1), \theta^{nN}]) &\leq V(t_1, u(t_1), v(t_1)) \leq V(t_0, u_0, v_0) \\ &\leq a(t_0, d_0[u_0, \theta^{nN}]) \leq a(t_0, \delta) \\ &< b(\varepsilon) \end{aligned}$$

this contradiction proves that (LS) holds.

In the case, if $g(t, V(t, u(t), v(t))) < 0$ (or $D^+V(t, u(t), v(t)) < -\beta V(t, u(t), v(t)))$ then we have

$$V(t, u(t), v(t)) \leq V(t_0, u_0, v_0),\ \forall t \geq t_0$$

and the trivial set solution is (LS). We need prove that

$$\lim_{t \to +\infty} d_0[u(t), \theta^{nN}] = 0.$$

We consider $D^+V(t, u(t), v(t)) < -\beta V(t, u(t), v(t))$ then

$$V(t, u(t), v(t)) \leq V(t_0, u_0, v_0).e^{[-\beta(t-t_0)]},\ \forall t \geq t_0.$$

If (ALS) is not holded, given ε_0, we choose $T = T(t_0, \varepsilon_0) = \dfrac{1}{\beta} \ln \dfrac{a(t_0, \delta)}{b(\varepsilon_0)} + 1$ then

$$b(d_0[u(t), \theta^{nN}]) \leq V(t, u(t), v(t)) \leq a(t_0, \delta).e^{[-\beta(t-t_0)]} \leq b(\varepsilon),\ \forall t \geq t_0 + T$$

this contradiction proves that (ALS) holds. □

Theorem 2. Assume that the positive Lyapunov - like function $V \in C[\mathbb{R}_+ \times E^{nN} \times E^{nP}, \mathbb{R}_+]$ satisfies the followings:

(i) $|V(t, \bar{u}(t), \bar{v}(t)) - V(t, u(t), v(t))| \leq L(d_0[\bar{u}(t), u(t)] + d_0[\bar{v}(t), v(t)])$, where L is bounded Lipschizt constant, for all $\bar{u}(t), u(t) \in E^{nN}, \bar{v}(t), v(t) \in E^{nP}$ and $t \in \mathbb{R}_+$;

(ii) $\exists \lambda_1(t), \lambda_2(t), \lambda_3(t), p, q > 0$ where $\lambda_1(t)$ increassing function such that

$$\lambda_1(t) d_0[u(t), \theta^{nN}]^p \leq V(t, u(t), v(t)) \leq \lambda_2(t) d_0[u(t), \theta^{nN}]^q;$$

(iii) $D^+V(t, u(t), v(t)) \leq -\lambda_3 d_0[u(t), \theta^{nN}]^s + K.e^{(-\delta t)}, \ \forall t > 0, \ u \in E^{nN} \setminus \{\theta^{nN}\}$;

(iv) $\delta > \inf\limits_{t \in \mathbb{R}_+} \left\{ \dfrac{\lambda_3(t)}{[\lambda_2(t)]^{l/q}} \right\} > 0$;

(v) $V(t, u(t), v(t)) - [V(t, u(t), v(t))]^{l/q} \leq \gamma e^{-\delta t}$ where $s, K, l, \gamma, \delta > 0$ then a trivial set solution of FCDE (3) is (ELS).

Proof. By (ii), we have

$$\lambda_1(t) d_0[u(t), \theta^{nN}]^p \leq V(t, u(t), v(t)) \Rightarrow d_0[u(t), \theta^{nN}] \leq \Big[\frac{V(t, u(t), v(t))}{\lambda_1(t)}\Big]^{1/p}.$$

Since $\lambda_1(t)$ nondecreasing on $t, \lambda_1(t_0) \leq \lambda_1(t), t \geq t_0$, we have

$$d_0[u(t), \theta^{nN}] \leq \Big[\frac{V(t, u(t), v(t))}{\lambda_1(t_0)}\Big]^{1/p}.$$

Let $M = \inf \left\{ \dfrac{\lambda_3(t)}{[\lambda_2(t)]^{l/q}} \right\}$ then $M < \delta$.
Set the function $Q(t, u(t), v(t)) = V(t, u(t), v(t)) e^{M(t-t_0)}$

$$\Rightarrow V(t, u(t), v(t)) = \frac{Q(t, u(t), v(t))}{e^{M(t-t_0)}}.$$

Replacing on

$$d_0[u(t), \theta^{nN}] \leq \left[\frac{Q(t, u(t), v(t))}{\lambda_1(t_0) e^{M(t-t_0)}}\right]^{1/p}. \tag{4}$$

On other hand, we compute

$$D^+Q(t, u(t), v(t)) = D^+V(t, u(t), v(t)) e^{M(t-t_0)} + MV(t, u(t), v(t)) e^{M(t-t_0)}.$$

By (ii) and $\lambda_2(t) > 0$ we have

$$d_0[u(t), \theta^{nN}]^q \geq \frac{V(t, u(t), v(t))}{\lambda_2(t)} \Leftrightarrow -d_0[u(t), \theta^{nN}]^l \geq -\Big[\frac{V(t, u(t), v(t))}{\lambda_2(t)}\Big]^{l/q}.$$

Therefore

$$\begin{aligned} D^+Q(t,u(t),v(t)) &\le \Big\{-[V(t,u(t),v(t))]^{l/q}\frac{\lambda_3(t)}{[\lambda_2(t)]^{l/q}}+Ke^{-\delta t}\Big\}e^{M(t-t_0)} \\ &\quad +MV(t,u(t),v(t))e^{M(t-t_0)}. \end{aligned}$$

Since $\dfrac{\lambda_3(t)}{[\lambda_2(t)]^{l/q}} \ge M,\ t\ge 0$ combine with (v), we have

$$\begin{aligned} D^+Q(t,u(t),v(t)) &\le M\{V(t,u(t),v(t))-[V(t,u(t),v(t))]^{l/q}\}e^{M(t-t_0)}+Ke^{-\delta t}e^{M(t-t_0)} \\ &\le M\gamma e^{-\delta t}e^{M(t-t_0)}+Ke^{-\delta t}e^{M(t-t_0)} \\ &= (M\gamma+K)e^{-\delta t}e^{M(t-t_0)} \\ &\le (M\gamma+K)e^{-\delta(t-t_0)}e^{M(t-t_0)} \end{aligned}$$

and we obtain

$$\begin{aligned} Q(t,u(t),v(t))-Q(t_0,u_0,v_0) &\le \int_{t_0}^{t}(M\gamma+K)e^{-\delta(s-t_0)}e^{M(s-t_0)}ds \\ &\le (M\gamma+K)\frac{1}{M-\delta}(e^{(M-\delta)(t-t_0)}-1). \end{aligned}$$

Setting $\delta_1=-(M-\delta)$ then $\delta_1>0$ by (iv)

$$\begin{aligned} Q(t,u(t),v(t)) &\le Q(t_0,u_0,v_0)+\frac{(M\gamma+K)}{\delta_1}-\frac{(M\gamma+K)}{\delta_1}e^{(M-\delta)(t-t_0)} \\ &\le Q(t_0,u_0,v_0)+\frac{(M\gamma+K)}{\delta_1}. \end{aligned}$$

Since $Q(t_0,u_0,v_0)=V(t_0,u_0,v_0)\le\lambda_2(t)d_0[u_0,\theta^{nN}]^q$, it follows that

$$Q(t,u(t),v(t))\le\lambda_2(t)d_0[u_0,\theta^{nN}]^q+\frac{(M\gamma+K)}{\delta_1}=\beta(d_0[u_0,\theta^{nN}],t_0).\ \forall t\ge t_0 \tag{5}$$

where $\beta(d_0[u_0,\theta^{nN}],t_0)=Q(t_0,u_0,v_0)+\dfrac{(M\gamma+K)}{\delta_1}>0$ combining (4) and (5),

$$d_0[u(t),\theta^{nN}]\le\Big\{\frac{\beta(d_0[u_0,\theta^{nN}],t_0)}{\lambda_1(t_0)}\Big\}^{1/p}.e^{-\frac{M(t-t_0)}{p}},\quad \forall t\ge t_0$$

so the trivial set solution of FCDE is (ELS). The proof is complete. □

Definition 3. The trivial set solution of FSCDE (3) is said to be

(S1) equi-stable of for each $\varepsilon>0$ and $t_0>0$, there exists a $\delta=\delta(t_0,\varepsilon)$ such that $d_0[u_0,\theta^{nN}]<\delta$ implies $d_0[u(t),\theta^{nN}]<\varepsilon$, for $t\ge t_0$;

(S2) uniformly stable, if the δ in (S1) is independent of t_0;

(S3) quasi-equi-asymptotically stable, if for each $\varepsilon > 0$, $t_0 > 0$, there exist a $T = T(t_0, \varepsilon)$ and $\delta_0 = \delta_0(t_0)$ such that $d_0[u_0, \theta^{nN}] < \delta_0$ implies $d_0[u(t), \theta^{nN}] < \varepsilon$, for all $t > t_0 + T$;

(S4) quasi-uniformly asymptotically stable, if δ_0 and T in (S3) are independent of t_0;

(S5) equi-asymptotically stable, if (S1) and (S3) hold simultaneously;

(S6) uniformly asymptotically stable, if (S2) and (S4) hold simultaneously;

(S7) exponentially asymptotically stable, if

$$d_0[u(t), \theta^{nN}] \leq \beta(d_0[u_0, \theta^{nN}], t_0) \exp[-\alpha(t - t_0)], t > t_0$$

where $\beta(d_0[.,.], t_0) : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$.

Remark 1. According to the definitions 2 and 3, we have:

(S1)$\Leftrightarrow$ (LS).

(S6)$\Leftrightarrow$ (ALS).

(S7)$\Leftrightarrow$ (ELS).

(S6)(or (ALS))$\Rightarrow$ (S3).

(S6)$\Rightarrow$ (S4).

We have to prove (S2) and (S6).

Theorem 3. Assume that the positive Lyapunov - like function $V \in C[\mathbb{R}_+ \times E^{nN} \times E^{nP}, \mathbb{R}_+]$ satisfies the followings:

(i) $|V(t, \bar{u}(t), \bar{v}(t)) - V(t, u(t), v(t))| \leq L(d_0[\bar{u}(t), u(t)] + d_0[\bar{v}(t), v(t)])$, where L is bounded Lipschizt constant, for all $\bar{u}(t), u(t) \in E^{nN}, \bar{v}(t), v(t) \in E^{nP}$ and $t \in \mathbb{R}_+$;

(ii) $b(d_0[u(t), \theta^{nN}] + d_0[v(t), \theta^{nP}]) \leq V(t, u(t), v(t)) \leq a(t, d_0[u(t), \theta^{nN}] - d_0[v(t), \theta^{nP}])$, for $(t, u(t), v(t)) \in \mathbb{R}_+ \times S^c(r) \times E^{nP}$ where $b(.), a(t, .)$ are increassing functions;

(iii)

$$\begin{aligned} D^+V(t, u(t), v(t)) \equiv & \lim_{\tau \to 0^+} \sup \frac{1}{\tau} \Big\{ V(t + \tau, u(t) + \tau f(t, u(t), v(t)), v(t) \\ & + \tau[\lambda(t) f(t, u(t), v(t)) + \lambda'(t) u(t)]) - V(t, u(t), v(t)) \Big\} \\ \leq & \; g(t, V(t, u(t), v(t))), \end{aligned}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}], g(t, 0) = 0$ for all $u(t) \in E^{nN}, v(t) = \lambda(t) u(t) \in E^{nP}$ and $t \in \mathbb{R}_+$.

Further more

a. If $g(t, V(t, u(t), v(t))) \leq 0, \forall t \geq t_0$ then (S2) holds.

b. If $g(t, V(t, u(t), v(t))) < 0, \forall t \geq t_0$ (or if $g(t, V(t, u(t), v(t))) < -c(d_0[u(t), \theta^{nN}]), \forall t \geq t_0$) then (S6) holds.

Proof. The condition (iii) with a/ (or b/) guarantees that

$$V(t, u(t), v(t))) \leq V(t_0, u(t_0), v(t_0))) \; \forall t \geq t_0.$$

a/ Let $\forall \varepsilon > 0, \forall t \geq t_0$. Choose $\delta = \delta(\varepsilon)$, such that

$$a(t_0, \delta) < b(\varepsilon) \text{ and } d_0[u(t_0), \theta^{nN}] \leq \delta$$

implies $d_0[u(t), \theta^{nN}] < \varepsilon$ that means (S2) holds.

If it is not true, then $\forall \delta > 0, \exists \varepsilon_0(\delta) > \delta$ and $a(t_0, \delta) < b(\varepsilon)$ such that $u(t) = u(t_0, u_0, t, v(t))$ is a set solution of FSCDE (3) according to control $v(t)$, which is satisfies $d_0[u(t), \theta^{nN}] \geq \varepsilon_0$. But on the other hands, we have

$$\begin{aligned} b(\delta) \leq b(\varepsilon_0) &= b(d_0[u(t), \theta^{nN}]) \leq b(d_0[u(t), \theta^{nN}] + d_0[v(t), \theta^{nP}]) \\ &\leq V(t, u(t), v(t)) \leq V(t_0, u_0, v_0) \leq a(t_0, (d_0[u(t_0), \theta^{nN}] - d_0[v(t_0), \theta^{nP}])) \\ &\leq a(t_0, d_0[u(t_0), \theta^{nN}]) = a(t_0, \delta) \leq b(\delta). \end{aligned}$$

This contradiction proves (S2).
b/ If $g(t, V(t, u(t), v(t))) < -c(d_0[u(t), \theta^{nN}]), \forall t \geq t_0$ we have (S2), that means

$$\forall \varepsilon > 0, \exists \delta(\varepsilon), \forall t_0 : d_0[u(t_0), \theta^{nN}] \leq \delta \text{ implies that } d_0[u(t), \theta^{nN}] \leq \varepsilon, \forall t \geq t_0.$$

Suppose that (S6) doesn't hold, that means $\forall \delta > 0, \exists \varepsilon(\delta) = \varepsilon_0 > \delta,\; \exists T = 1 + \dfrac{a(t_0, \delta)}{c.\varepsilon_0}$ such that

$$d_0[u(t), \theta^{nN}] \geq \varepsilon_0, \forall t \in [t_0, t_0 + T].$$

Because $g(t, V(t, u(t), v(t))) < -c(d_0[u(t), \theta^{nN}]), \forall t \geq t_0$ then

$$V(t, u(t), v(t)) \leq V(t_0, u(t_0), v(t_0)) - c \int_{t_0}^{t} d_0[u(t), \theta^{nN}] dt, \forall t \in [t_0, t_0 + T]$$

implies

$$0 \leq V(t_0 + T, u(t_0 + T), v(t_0 + T)) \leq a(t_0, \delta) - c.\varepsilon_0.T < 0.$$

This contradiction proves (S6). □

Corollary 1. the function $g(t, V(t, u, v)) \equiv 0$ is admissible in Theorem 3 then for everry $\alpha > 0,\; t_0 > 0,\; \exists \beta(t, \alpha) > 0$ such that

$$d_0[u_0, \theta^{nN}] < \alpha \text{ implies } d_0[u(t), \theta^{nN}] < \beta,\; \forall t > t_0.$$

3.2. Problem Of Controllability

Anologously (Phu N. D., 2009), we consider the problem (GC) for fuzzy set control differential equations.

Definition 4. The fuzzy set control differential equation (3) is said to be

(GC) Global controllable if every state pair of set solutions $(u_0, \bar{u}) \in E^{nN}$ will be a controllable;

(GA) Global achievable if every $\bar{u} \in E^{nN}$ we have state pair of set solutions $(\theta^{nN}, \bar{u}) \in E^n$ will be a controllable;

(GAZ) Global achievable to zero set point $\theta^{nN} \in E^{nN}$ if every $\bar{u} \in E^{nN}$ we have state pair of set solutions $(\bar{u}, \theta^{nN}) \in E^{nN}$ will be a controllable.

Suppose that, the set global achievable (GA) from $u_0 \in E^{nN}$ after time t:

(i) $R_t(u_0) = \Big\{ u(t) \in E^{nN} \,|\exists v(t) \in E^{nP} : u(t_0, u_0, t, v(t)) = v(t) \Big\};$

(ii) $R(u_0) = \bigcup_{t>0} R_t(u_0).$

According to definition 4, we have:

Theorem 4. (i) The fuzzy set control differential equation (3) is (GC) if for every $u_0 \in E^{nN}$ we have $R(u_0) = E^{nN}$;

(ii) The fuzzy set control differential equation (3) is (GA) if we have $R(\theta^{nN}) = E^{nN}$;

(iii) The fuzzy set control differential equation (3) is (GAZ) if for every $u_0 \in E^{nN}$ we have $\theta^{nN} \in R(u_0)$.

In the following we consider many kinds of SFCDE :

A. Stationary linear fuzzy set control differential equation (SLFSCDE):

We consider SLFSCDE:

$$D_{Ht} u(t) = Au(t) + Bv(t), \quad u(t_0) = u_0 \in E^{nN} \tag{6}$$

where A, B are operators: $A : E^{nN} \to E^{nN}$, $\;B : E^{nP} \to E^{nN}$, $t \in I \subset \mathbb{R}_+$, $v(t) \in E^{nP}$ whose fuzzy set solution is form $u(t) = u(t_0, u_0, t, v(t)) \in E^{nN}$. Let the Cauchy operator for $D_{Ht}u(t) = Au(t)$ is: $W_0(t, s) = W_0(t, \tau)W_0(\tau, s)$ then set solution of SLFSCDE (6) is rewritted in the form:

$$u(t) = u(t_0, u_0, t, v(t)) = W_0(t, t_0)u_0 + \int_{t_0}^{t} W_0(t, s)Bv(s)ds \tag{7}$$

If we put $L_t = \int_{t_0}^{t} W_0(t, s)W_0(t, t_0)ds$ then $L_{t_1} = \int_{t_0}^{t_1} W_0(t_1, s)W_0(t_1, t_0)ds$

Theorem 5. The SLFSCDE (6) is (GC) if and only if exist B^{-1} and $L_{t_1}^{-1}$.

Proof. a/ Sufficient condition: By $L_t = \int_{t_0}^{t} W_0(t,s)W_0(t,t_0)ds$, for every $t_1 > t_0$ we find the fuzzy feedback $v(t) \in E^{nP}$ in type:

$$v(s) = -B^{-1}W_0(t_1,s)L_{t_1}^{-1}[W_0(t_1,t_0)u_0 - \bar{u}] \tag{8}$$

Replacing this fuzzy feedback (8) into (7) at moment $t_1 > t_0$, we have

$$\begin{aligned}
u(t_1) &= X(t_0,u_0,t_1,v(t_1)) = W_0(t_1,t_0)u_0 + \int_{t_0}^{t_1} W_0(t,s)Bv(s)ds \\
&= W_0(t_1,t_0)u_0 - \int_{t_0}^{t_1} W_0(t,s)B\{B^{-1}W_0(t_1,s)L_{t_1}^{-1}[W_0(t_1,t_0)u_0 - \bar{u}]\}ds \\
&= W_0(t_1,t_0)u_0 - \int_{t_0}^{t_1} W_0(t,s)BB^{-1}W_0(t_1,s)L_{t_1}^{-1}[W_0(t_1,t_0)u_0 - \bar{u}]ds \\
&= W_0(t_1,t_0)u_0 - L_{t_1}L_{t_1}^{-1}[W_0(t_1,t_0)u_0 - \bar{u}] \\
&= W_0(t_1,t_0)u_0 - [W_0(t_1,t_0)u_0 - \bar{u}] = \bar{u}
\end{aligned}$$

That means the state pair $(u_0, \bar{u}) \in E^n$ will be controllable or SLFSCDE (6) is (GC).
b/ Necessary condition: Let SLFSCDE (6) is (GC) we can poof that exist B^{-1} and $L_{t_1}^{-1}$. The necessity of Theorem 5 is obvious. If given operator B has not B^{-1} that means B is decline then SLFSCDE (6) isn't controllabe. If $L_{t_1}^{-1}$ isn't existed then didn't exist control (8), so that SLFSCDE (6) isn't (GC). □

B. Stationary linear fuzzy set control differential equation with pertubation (SLF-SCDEP):

We consider SLFSCDEP:

$$D_{Ht}u(t) = Au(t) + Bv(t) + R(t,u(t),v(t)), \quad u(t_0) = u_0 \in E^{nN} \tag{9}$$

where A, B are operators: $A : E^{nN} \to E^{nN}$, $B : E^{nP} \to E^{nN}$, $t \in I \subset \mathbb{R}_+$, $v(t) \in E^{nP}$ whose fuzzy set solution is form $u(t) = u(t_0,u_0,t,v(t)) \in E^{nN}$. Let the Cauchy operator for $D_{Ht}u(t) = Au(t)$ is: $W_0(t,s) = W_0(t,\tau)W_0(\tau,s)$ then fuzzy set solution of SLFSCDEP (9) is rewritted in the form:

$$u(t) = u(t_0,u_0,t,v(t)) = W_0(t,t_0)u_0 + \int_{t_0}^{t} W_0(t,s)Bv(s)ds + \int_{t_0}^{t} R(t,u(s),v(s))ds$$

If we put $L_t = \int_{t_0}^{t} W_0(t,s)W_0(t,t_0)ds$ then $L_{t_1} = \int_{t_0}^{t_1} W_0(t_1,s)W_0(t_1,t_0)ds$.
Suppose

$$\lim_{\|u\|\to 0} \frac{\|R(t,u,v)\|}{\|u\|} = 0. \tag{10}$$

Theorem 6. The SLFSCDEP (9) with (10) is (GC) if and only if exist B^{-1} and $L_{t_1}^{-1}$.

C. Unstationary linear fuzzy set control differential equation (USLFSCDE):

We consider USLFSCDE:

$$D_{Ht}u(t) = A(t)u(t) + B(t)v(t), \quad u(t_0) = u_0 \in E^{nN} \tag{11}$$

where A, B are operators, which defend on time $t, t \in I \subset \mathbb{R}_+, v(t) \in E^p$ whose set solution is form $u(t) = u(t_0, u_0, t, v(t)) \in E^{nN}$. Let the Cauchy operator for $D_{Ht}u(t) = A(t)u(t)$ is: $W_1(t,s) = W_1(t,\tau)W_1(\tau,s)$ then set solution of USLFSCDE (11) is rewritted in the form:

$$u(t) = u(t_0, u_0, t, v(t)) = W_1(t, t_0)u_0 + \int_{t_0}^{t} W_1(t,s)B(s)v(s)ds \tag{12}$$

If we put $\Lambda_t = \int_{t_0}^{t} W_1(t,s)W_1(t,t_0)ds$ then $\Lambda_{t_1} = \int_{t_0}^{t_1} W_1(t_1,s)W_1(t_1,t_0)ds$.

Theorem 7. The USLFSCDE (11) is (GC) if and only if exist $B^{-1}(s)$ and $\Lambda_{t_1}^{-1}$.

Proof. a/ Sufficient condition: By $\Lambda_{t_1} = \int_{t_0}^{t_1} W_1(t_1,s)W_1(t_1,t_0)ds$, for every $t_1 > t_0$ we find the fuzzy feedback $v(t) \in E^{nP}$ in type:

$$v(s) = -B^{-1}(s)W_1(t,s)\Lambda_{t_1}^{-1}[W_1(t,t_0)u_0 - \bar{u}] \tag{13}$$

Replacing this fuzzy feedback (13) in (12) at moment $t_1 > t_0$, we have

$$\begin{aligned}
u(t_1) &= u(t_0, u_0, t_1, v(t_1)) = W_1(t_1,t_0)u_0 + \int_{t_0}^{t_1} W_1(t_1,s)B(s)v(s)ds \\
&= W_1(t_1,t_0)u_0 - \int_{t_0}^{t_1} W_1(t_1,s)B(s)\{B^{-1}(s)W_1(t_1,s)\Lambda_{t_1}^{-1}[W_1(t_1,t_0)u_0 - \bar{u}]\}ds \\
&= W_1(t_1,t_0)u_0 - \int_{t_0}^{t_1} W_1(t_1,s)B(s)B^{-1}(s)W_1(t_1,s)\Lambda_{t_1}^{-1}[W_1(t_1,t_0)u_0 - \bar{u}]ds \\
&= W_1(t_1,t_0)u_0 - \Lambda_{t_1}\Lambda_{t_1}^{-1}[W_1(t_1,t_0)u_0 - \bar{u}] \\
&= W_1(t_1,t_0)u_0 - [W_1(t_1,t_0)u_0 - \bar{u}] = \bar{u}.
\end{aligned}$$

That means the state pair $(u_0, \bar{u}) \in E^n$ will be controllable or USLFSCDE (11) is (GC).
b/ Necessary condition: Let USLFSCDE (11) is (GC) we can poof that exist $B^{-1}(s)$ and $\Lambda_{t_1}^{-1}$. The necessity of Theorem 3.7 is obvious. If given operator $B(s)$ has not $B^{-1}(s)$ that means $B(t)$ is decline then USLFSCDE (11) isn't controllabe. If $\Lambda_{t_1}^{-1}$ isn't existed then didn't exist control (13), so that USLFSCDE (11) isn't (GC). □

D. Unstationary linear fuzzy set control differential equation with perturbation (USLF-SCDEP).

We consider USLFSCDEP:

$$D_{Ht}u(t) = A(t)u(t) + B(t)v(t) + R(t, u(t), v(t)), \quad u(t_0) = u_0 \in E^{nN} \tag{14}$$

where $A(t), B(t)$ are operators, which defend on time, $t \in I \subset \mathbb{R}_+, v(t) \in E^{nP}$ and $R(t, u(t), v(t))$ is perturbationing set.

The set solution of (14) is form $u(t) = u(t_0, u_0, t, v(t)) \in E^{nN}$. Let the Cauchy operator for USLFSCDE $D_{Ht}u(t) = A(t)u(t)$ is: $W_1(t,s) = W_1(t,\tau)W_1(\tau,s)$, then set solution of USLFSCDEP (14) is rewritted in the form:

$$u(t) = u(t_0, u_0, t, v(t)) = W_1(t,t_0)u_0 + \int_{t_0}^{t} W_1(t,s)B(s)v(s)ds + \int_{t_0}^{t} R(t, u(s), v(s))ds$$

Theorem 8. The USLFSCDEP (14) with (10) is (GC) iff exist $B^{-1}(s)$ and $\Lambda_{t_1}^{-1}$.

E. Nonlinear fuzzy set control differential equation (NLFSCDE).
We consider NLFSCDE :

$$D_{Ht}u(t) = f(t, u(t), v(t)), \quad u(t_0) = u_0 \in E^{nN} \tag{15}$$

where $f \in C[I \times E^{nN} \times E^{nP}, E^{nN}], t \in I \subset \mathbb{R}_+, v(t) \in E^{nP}$.
The set solution of NLFSCDE (15) is :

$$u(t) = u(t_0, u_0, t, v(t)) = u_0 + \int_{t_0}^{t} f(s, u(s), v(s))ds.$$

Theorem 9. Suppose that for NLFSCDE (15) satisfy the followings:

(i) $f \in C[I \times E^{nN} \times E^{nP}, E^{nN}],\ \|f(t, u(t), v(t))\| \leq \bar{L}(\|u\| + \|v\|), \bar{L} > 0;$

(ii) For every pair of fuzzy set states $(u_0, \bar{u}) \in E^{nN}, \forall t_1 > t_0, \exists v(t) \in E^{nP}$ such that

$$\int_{t_0}^{t_1} \|v(s)\|ds = \frac{\|\bar{u}\| \exp[-\bar{L}(t_1 - t_0)] - \|u_0\|}{\bar{L}} \tag{16}$$

then NLFSCDE (15) is (GC).

Proof. Before we choose the controllable set $v(t) \in E^{nP}$ satisfied (16).
Because the set solution of (15) is writted in the form

$$u(t) = u(t_0, u_0, t, v(t)) = u_0 + \int_{t_0}^{t} f(s, u(s), v(s))ds.$$

then $u(t_1) = u(t_0, u_0, t_1, v(t_1)) = u_0 + \int_{t_0}^{t_1} f(s, u(s), v(s))ds$ implies that

$$\begin{aligned} \|u(t_1)\| &\leq \|u_0\| + \int_{t_0}^{t_1} \|f(s, u(s), v(s))\|ds \\ &\leq \|u_0\| + \int_{t_0}^{t_1} \bar{L}(\|u(s)\| + \|v(s)\|)ds \\ &\leq \|u_0\| + \bar{L}\int_{t_0}^{t_1} \|u(s)\|ds + \bar{L}\int_{t_0}^{t_1} \|v(s)\|ds \end{aligned} \tag{17}$$

Replacing

$$\int_{t_0}^{t_1} \|v(s)\|ds = \frac{\|\bar{u}\| \exp[-\bar{L}(t_1 - t_0)] - \|u_0\|}{\bar{L}}$$

from (16) into (17) we received $\|u(t_1)\| \leq \|\bar{u}\|$.The proof is complete, if we prove that $\|u(t_1)\| = \|\bar{u}\|$. If this is not true, that if $\|u(t_1)\| < \|\bar{u}\|$. Because $\bar{u} \in E^{nN}$ - one of the fuzzy set solutions NLFSCDE, then for admissible control $v(t)$ we have

$$\bar{u} = \bar{u}(t_1) = u_0 + \int_{t_0}^{t_1} [f(s, \bar{u}(s), v(s))]ds.$$

We have to prove that $\|u(t_1)\| = \|\bar{u}\|$. In other hand, we have

$$\begin{aligned} d_0[u(t_1), \bar{u}(t_1)] &\leq d_0\Big[\int_{t_0}^{t_1} f(s, u(s), v(s))ds, \int_{t_0}^{t_1} f(s, \bar{u}(s), v(s))ds\Big] \\ &\leq \int_{t_0}^{t_1} d_0[f(s, u(s), v(s)), f(s, \bar{u}(s), v(s))]ds \\ &\leq \bar{L}\int_{t_0}^{t_1} d_0[u(s), \bar{u}(s)]ds. \end{aligned}$$

Putting $d_0[u(t_1), \bar{u}(t_1)] = p(t_1)$ we have $p(t_1) \leq \int_{t_0}^{t_1} p(s)ds$. By the Gronwall-Bellman's lemma, we have $p(t_1) = 0$. That means $\|u(t_1)\| = \|\bar{u}\|$ and the state pair $(u_0, \bar{u}) \in E^{nN}$ will be controllable, the NLFSCDE (15) is (GC). □

Theorem 10. Suppose that for NLFSCDE (15) satisfy the followings:

(i) $f \in C[I \times E^n \times E^p, E^n],\ \|f(t, u(t), v(t))\| \leq \bar{L}(\|u\| + \|v\|), \bar{L} > 0$;

(ii) For every pair of set states $(u_0, \bar{u}) \in E^{nN}, \forall t_1 > t_0, \exists v(t) \in E^{nP}$ such that $v(t) = \lambda(t)\bar{u}$, where the feedback operator $\lambda(t)$ satisfies:

$$\int_{t_0}^{t_1} \|\lambda(s)\|ds = \frac{\|\bar{u}\| \exp[-\bar{L}(t_1 - t_0)] - \|u_0\|}{\bar{L}\|\bar{u}\|} \tag{18}$$

then NLFSCDE (15) is (GC).

Proof. Before we choose the feedback set: $v(t) = \lambda(t)u_1$, whose $\lambda(t)$ satisfies (18).
Because the set solution of (15) is

$$u(t) = u(t_0, u_0, t, v(t)) = u_0 + \int_{t_0}^{t} f(s, u(s), v(s))ds.$$

then

$$u(t_1) = u(t_0, u_0, t_1, v(t_1)) = u_0 + \int_{t_0}^{t_1} f(s, u(s), v(s))ds$$

implies that

$$\begin{aligned} \|u(t_1)\| &\leq \|u_0\| + \int_{t_0}^{t_1} \|f(s, u(s), v(s))\|ds \\ &\leq \|u_0\| + \int_{t_0}^{t_1} \bar{L}(\|u(s)\| + \|v(s)\|)ds \\ &\leq \|u_0\| + \bar{L}\int_{t_0}^{t_1} \|u(s)\|ds + \bar{L}\int_{t_0}^{t_1} \|v(s)\|ds \end{aligned} \tag{19}$$

Replacing

$$\int_{t_0}^{t_1} \|\lambda(s)\|ds = \frac{\|\bar{u}\| \exp[-\bar{L}(t_1 - t_0)] - \|u_0\|}{\bar{L}\|\bar{u}\|}$$

from (18) into (19) we received $\|u(t_1)\| \leq \|\bar{u}\|$.
The proof is complete, if we prove that $\|u(t_1)\| = \|\bar{u}\|$. If this is not true, that if $\|u(t_1)\| < \|\bar{u}\|$. We have partially similar proof of theorem 9, and finally $\|u(t_1)\| = \|\bar{u}\|$. That means the state pair $(u_0, \bar{u}) \in E^{nN}$ will be controllable, the NLFSCDE (15) is (GC). □

Suppose that the fuzzy mapping set $f(t, u(t), v(t))$ in NLFSCDE (15) is differentiable:

Corollary 2. In the case, when $P = N$ we must choose the feedback operator $\lambda(t)$ as real function such that for every pair of set states $(u_0, \bar{u}) \in E^{nN}, \forall t_1 > t_0, \exists v(t) \in E^{nN}$ with $v(t) = \lambda(t)X_1$, where this function $\lambda(t)$ satisfies:

$$\int_{t_0}^{t_1} |\lambda(s)| ds = \frac{\|\bar{u}\| \exp[-\bar{L}(t_1 - t_0)] - \|u_0\|}{\bar{L}\|\bar{u}\|}$$

then NLFSCDE (15) is (GC).

Corollary 3. (i) If fuzzy mapping set $f(t, u(t), v(t))$ is differentiable (see (Phu and Dung, to appear)) and it's approximation is $Au + Bv + R(t, u, v)$ then problem (GC) of NLFSCDE (15) will be a problem (GC) of SLFSCDEP (9).

(ii) If fuzzy mapping set $f(t, u(t), v(t))$ is differentiable (see (Phu and Dung, to appear)) and it's approximation is $A(t)u + B(t)v + R(t, u, v)$ then problem (GC) of NLFSCDE (15) will be a problem (GC) of UNSLFSCDEP (14).

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