A novel risk definition for portfolio selection with uncertain returns

Xiaoxia Huang
School of Economics and Management
University of Science and Technology Beijing, Beijing 100083, China
hxiaoxia@manage.ustb.edu.cn

Abstract. Portfolio selection is concerned with optimization of capital allocation to a large number of securities. In portfolio selection, risk analysis is one of the most important topics and research on quantitative definition of risk remains core of the topic. This paper proposes a novel risk definition for portfolio selection with uncertain returns. A risk curve is introduced and a new safe criterion for judging the portfolio investment is proposed.

Keywords: risk curve, portfolio selection, uncertainty, uncertain variable

1. Introduction

Portfolio selection is concerned with optimization of capital allocation to a large number of securities. Traditionally, security returns were regarded to be random variables, and probability theory is the main tool to deal with portfolio selection problem. Since the security market is complex, it is found that future security returns are hard to be always well reflected by the historical data. With the introduction of fuzzy set theory by Zadeh (1965), scholars began to realize that they could employ fuzzy set theory to manage portfolio in another type of uncertain environment called fuzzy environment. In the area of fuzzy portfolio selection, Huang (2007) used credibility measure to develop a credibilistic mean-variance methodology, VaR (2006) method, and mean-risk model (2008a). Some researchers such as Bilbao-Terol et al. (2006), Lacagnina and Pecorella (2006) and Vercher (2008), etc. employed possibility measure to study fuzzy portfolio selection problems.

Though fuzzy set theory and credibility theory provide alternative tools for handling portfolio selection with returns other than randomness, there are other types of uncertainty in the security market. Suppose a security return is likely to be about 0.1. The occurrence chance of the security return between 0.1 and 0.2 is 30%, and the occurrence chance of the security return between 0.2 and 0.3 is 20% (see Fig. 1). Then what do you think the occurrence chance of the security return between 0.1 and 0.3 to be? If you think the occurrence chance will be 50%, you in fact are believing that the security return can be described by random variable; if you think the occurrence chance will be 30%, you in fact are believing that the security return should be described by fuzzy variable. However, if you think the occurrence chance should not be as big as 50%, nor should it be as small as 30%, instead, it should be a number between 30% and 50%, then you in fact are believing that the security return can be described by another kind of variable.

Recently, Liu (2007) studied the uncertainty other than randomness and fuzziness and proposed an uncertain measure and uncertain variable to describe the above mentioned new type of uncertainty which is neither random nor fuzzy. Furthermore, an uncertainty theory was developed and

4th International Workshop on Reliable Engineering Computing (REC 2010)
Edited by Michael Beer, Rafi L. Muhanna and Robert L. Mullen

ISBN: 978-981-08-5118-7. *Published by* Research Publishing Services.
doi:10.3850/978-981-08-5118-7_057

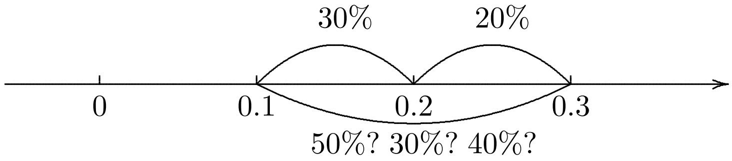

Figure 1. Security return may behave neither randomly nor fuzzily.

much research work has been done on the development of uncertainty theory and related theoretical work. For example, Liu discussed uncertain calculus (Liu, 2008) and uncertain programming (Liu, 2009). Gao (2009) discussed some properties of continuous uncertain measure. You (2009) proved some convergence theorems of uncertain sequences. Liu (2008) defined uncertain process, and Li and Liu (2009) discussed uncertain logic, etc. In (Qin et al., technique report), Qin, Kar and Li developed an uncertain mean-variance portfolio selection model. In this paper, we try to discuss the problem that if security returns are uncertain variables rather than random variable or fuzzy variable, how should we select the optimal portfolio? We try to extend the risk curve idea in paper (Huang, 2008b) to the new uncertain environment to solve the problem.

The rest of the paper is organized as follows. Why risk curve is necessary is discussed first in Section 2. Then, for better understanding of our extension of risk curve to new uncertain environment, some necessary knowledge about uncertain measure and uncertainty theory will be reviewed in Section 3. After that, uncertainty risk curve and confidence curve will be introduced in Sections 4 and 5, respectively, and a mean-risk model for portfolio selection with uncertain returns will be developed in Section 6. Finally, in Section 7, some conclusion remarks will be given.

2. Why Risk Curve

In portfolio selection, how to define risk is one of the most important topics. The earliest and the most popular risk definition is variance. It was given by Markowitz (1952) in 1952. He proposed that expected value of a portfolio return could be regarded as the representative of the investment return and variance the risk of the investment. The idea is that the greater deviation from the expected value, the less likely the investors can obtain the expected return. Thus, the risk of the investment is greater. However, variance may not be reasonable sometimes. For example, suppose we have two portfolios A and B. They may randomly take two return values respectively. The likely returns of the two portfolios are given in Table I. It is easy to see that the variance values of portfolios A and B are same. However, Portfolio B is much safer than portfolio A. In addition, in reality, people are not always concerned about average value. Sometimes, they are sensitive to the specific low return value that they may suffer. The popular Value-at-Risk (VaR) reflects the people's concern. However, VaR only provides information about one low return value that the investors may suffer. Other low return case may also happen. And once a very low return case (i.e., a huge loss event) happens, the investors may not be able to bear it. This is also true in case of uncertain portfolio investment when portfolio return is neither random nor fuzzy. Thus, a new risk measure that provides information about all the likely losses should be given. For better understanding of

Table I. Likely Returns of Portfolios A and B

Portfolio	Likely Return	Likely Return
A	0	1
B	100	101

the new risk measure, let us first review some necessary knowledge about uncertainty theory that will be used in the paper before giving the new risk definition.

3. Necessary Knowledge about Uncertain Variable

To describe the uncertain phenomenon mentioned in Introduction which is neither random nor fuzzy, Liu (2007) founded an uncertainty theory which is an axiomatic system of normality, monotonicity, self-duality, countable subadditivity and product measure.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}\{\Lambda\}$ is called an uncertain measure if it satisfies the following four axioms:

(i) (Normality) $\mathcal{M}\{\Gamma\} = 1$.

(ii) (Monotonicity) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_1\}$ whenever $\Lambda_1 \subset \Lambda_2$.

(iii) (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$.

(iv) (Countable Subadditivity) For every countable sequence of events$\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 1 (Liu, 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

In application, a random variable is usually characterized by a probability density function or probability distribution function. Similarly, an uncertain variable can be characterized by an uncertainty distribution function.

Definition 2 (Liu, 2007) The uncertainty distribution $\Phi : \Re \to [0, 1]$ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}.$$

For example, by a normal uncertain variable, we mean the variable that has the following normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re,$$

where μ and σ are real numbers and $\sigma > 0$. For convenience, it is denoted in the paper by $\xi \sim \mathcal{N}(\mu, \sigma)$.

We call an uncertain variable the linear uncertain variable if it has the following linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x < a \\ (x-a)/(b-a), & \text{if} \quad a \le x \le b \\ 1, & \text{if} \quad x > b. \end{cases}$$

For convenience, it is denoted in the paper by $\xi \sim \mathcal{L}(a, b)$ where $a < b$.

The operational law of uncertain variables is given by Liu (2008) as follows:

Theorem 1 (Liu, 2008) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables, and $f : \Re^n \to \Re$ a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable such that

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B} \min\limits_{1\le k\le n} \mathcal{M}\{\xi_k \in B_k\}, \\ \qquad \text{if} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B} \min\limits_{1\le k\le n} \mathcal{M}\{\xi_k \in B_k\} > 0.5 \\ 1 - \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B^c} \min\limits_{1\le k\le n} \mathcal{M}\{\xi_k \in B_k\}, \\ \qquad \text{if} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B^c} \min\limits_{1\le k\le n} \mathcal{M}\{\xi_k \in B_k\} \ge 0.5 \\ 0.5, \qquad \text{otherwise} \end{cases} \tag{1}$$

where $B, B_1, B_2, \cdots, B_n$ are Borel sets of real numbers.

Theorem 2 (Liu, Uncertainty Theory on Line) Let ξ be an uncertain variable with uncertainty distribution Φ, and let f be a strictly increasing function. Then the uncertainty distribution of $f(\xi)$ can be obtained via

$$\Psi(t) = \Phi(f^{-1}(t)). \tag{2}$$

Theorem 3 (Peng, technique report) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and Ψ the uncertainty distribution of the sum $\xi_1 + \xi_2 + \cdots + \xi_n$. If $\Phi_1^{-1}(\alpha)$, $\Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)$ are unique for each $\alpha \in (0, 1)$, we have

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha), \quad 0 < \alpha < 1. \tag{3}$$

To tell the size of an uncertain variable, Liu defined the expected value of uncertain variables.

Definition 3 (Liu, 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_0^{\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\}\mathrm{d}r \tag{4}$$

provided that at least one of the two integrals is finite.

It can be calculated that the expected value of the normal uncertain variable $\xi \sim \mathcal{N}(\mu, \sigma)$ is $\mathrm{E}[\xi] = \mu$, and the expected value of the linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is $\mathrm{E}[\xi] = (a + b)/2$.

Theorem 4 (Liu, Uncertainty Theory on Line) Let ξ_1 and ξ_2 be independent uncertain variables with finite expected values. Then for any real numbers a_1 and a_2, we have

$$\mathrm{E}[a_1\xi_1 + a_2\xi_2] = a_1\mathrm{E}[\xi_1] + a_2\mathrm{E}[\xi_2]. \tag{5}$$

4. Risk Curve

To give a new risk measure that provides information about all the likely losses in a portfolio investment with uncertain return, we should first answer the question "how should we describe all the likely losses mathematically"? Usually, when an investment return is -0.1, people will instinctively feel that they suffer a loss of 0.1. This in fact implies that people set their breakeven point at 0 and they experience a difference, i.e., $0 - (-0.1)$, of the investment return from the point. In portfolio investment, since the portfolio return is variable and may be -0.05, -0.11, $\cdots$, etc., people' loss may be 0.05, 0.11, $\cdots$, etc. When the portfolio return is 0.1, people will feel they gain and now the difference, i.e., $0-0.1$, of the portfolio return from the breakeven point is a negative number. Thus, it is clear that if ξ represents the uncertain portfolio return, then $0-\xi$ describes all the likely losses when $0 - \xi \geq 0$. Of course, the investors can set their breakeven point higher than 0. In financial investment, people have a choice to invest their money in risk-free asset and gain a return rate as high as the risk-free interest rate with certainty. Thus, the risk-free interest rate, denoted by r_f, can be chosen as the breakeven point in portfolio investment. Then, if the risk-free interest rate is 0.015, the investors will still suffer a loss of $0.015 - 0.01 = 0.005$ even when the portfolio return is 0.01. Now it is clear that the expression $r_f - \xi \geq r, \forall r \geq 0$ describes all the likely losses in one investment. Taking into account all the likely losses of an uncertain portfolio investment and the corresponding occurrence chances of these losses, we define the risk curve as follows:

Definition 4 Let ξ denote an uncertain return rate of a security, and r_f the risk-free interest rate. Then the curve

$$R(r) = \mathcal{M}\{r_f - \xi \geq r\}, \quad \forall r \geq 0 \tag{6}$$

is called the risk curve of the security.

It is clear that the risk curve can be expressed in the form $R(r) = \mathcal{M}\{\xi \leq r_f - r\}$. Thus, if we have the uncertainty distribution function of the security, we have the risk curve of the security. Furthermore, we know from the monotonicity axiom of the uncertain measure that $R(r)$ is a decreasing function of the real numbers r. That is, when the loss becomes bigger, the occurrence chance of the loss will become smaller. With the risk curve definition (6), given a loss level, the investors are able to know how high chance the loss may occur.

Equivalently, the risk curve can also be expressed in the form

$$R^{-1}(\alpha) = r_f - \Phi^{-1}(\alpha), \quad \forall \alpha \in (0, 1) \tag{7}$$

where Φ is the uncertainty distribution of ξ.

With the risk curve formulation (7), given a confidence level, the investors are able to know how much they will lose at this occurrence chance level.

For example, if the security return is a normal uncertain variable $\xi \sim \mathcal{N}(\mu, \sigma)$. Then the risk curve of the security is as follows (also see Fig. 2):

$$R(r) = \mathcal{M}\{r_f - \xi \geq r\} = \left(1 + \exp\left(\frac{\pi(\mu - r_f + r)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad \forall r \geq 0.$$

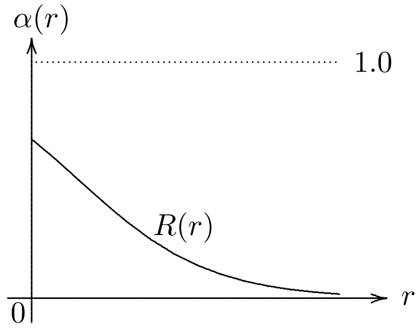

Figure 2. Risk curve of a security with normal uncertain return

If the security return is a linear uncertain variable $\xi \sim \mathcal{L}(a, b)$, the risk curve of the security is as follows (also see Fig. 3):

$$R(r) = \mathcal{M}\{(r_f - \xi) \geq r\} = \begin{cases} 1, & \text{if } r < r_f - b \\ \dfrac{r_f - a - r}{b - a}, & \text{if } r_f - b \leq r \leq r_f - a \\ 0, & \text{if } r > r_f - a. \end{cases}$$

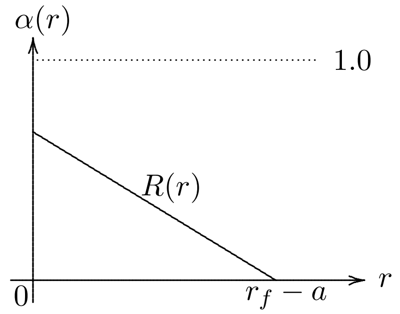

Figure 3. Risk curve of a security with linear uncertain return

5. Confidence Curve

Since all investors know that they may lose as well as gain in investment, they will have a maximum tolerance towards occurrence chance of each likely loss level. We call it confidence curve $\alpha(r)$ that gives the investors' maximal tolerance towards the occurrence chance of each likely loss level. Though different investors have different confidence curve, the common trend of the curve is that the severer the loss (i.e., the greater the r value), the lower tolerance level the investors have towards the occurrence chance of the loss. The general trend of the confidence curve is given in Fig.4.

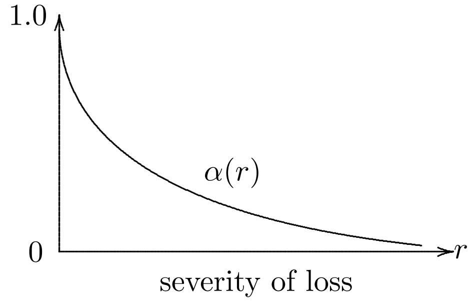

Figure 4. General trend of a confidence curve: The higher the loss value, the lower the tolerance of the occurrence chance of the loss.

6. Mean-Risk Model for Uncertain Portfolio Selection

With the quantitative definition of risk curve and the subjunctive standard of confidence curve, we are now able to judge if a portfolio investment is safe or not. Let x_i denote the investment proportions in securities $i, i = 1, 2, \cdots, n$, respectively, ξ_i the i-th security returns which are uncertain variables, and $\alpha(r)$ the investor's confidence curve. We say a portfolio is safe if

$$\mathcal{M}\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq r\} \leq \alpha(r), \quad \forall r \geq 0 \tag{8}$$

where r_f is the risk-free interest rate.

In portfolio investment, people will always ask that the investment should be safe enough. Then they will pursue the maximum return. Using expected value as the representative of the investment return, we build the mean-risk model to express the idea mathematically as follows:

$$\begin{cases} \max E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{subject to:} \\ \qquad R(x_1, x_2, \cdots, x_n; r) \leq \alpha(r), \, \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{9}$$

where E is the expected value operator defined by Equation (4), $\alpha(r)$ the investor's confidence curve, and $R(x_1, x_2, \cdots, x_n; r)$ the risk curve defined as

$$R(x_1, x_2, \cdots, x_n; r) = \mathcal{M}\left\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq r\right\}.$$

Theorem 5 Let Φ_i denote the uncertainty distributions of the i-th security return rates $\xi_i, i = 1, 2, \cdots, n$, respectively. Then the mean-risk model (9) can be transformed into the following linear model:

$$\begin{cases} \max x_1 \mathrm{E}[\xi_1] + x_2 \mathrm{E}[\xi_2] + \cdots + x_n \mathrm{E}[\xi_n] \\ \text{subject to:} \\ \qquad x_1 \Phi_1^{-1}\Big(\alpha(r)\Big) + x_2 \Phi_2^{-1}\Big(\alpha(r)\Big) + \cdots + x_n \Phi_n^{-1}\Big(\alpha(r)\Big) \geq r_f - r, \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{10}$$

Proof: It follows from Theorem 4 that the objective function of model (9) can be transformed into the objective function of model (10).

It follows from Equation (2) that if ξ is an uncertain variable with uncertainty distribution Φ. Then for any number $k > 0$, the uncertainty distribution of $k\xi$

$$\Psi(t) = \Phi\left(\frac{t}{k}\right) \quad \text{and} \quad \Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha).$$

Thus, according to Equation (3), the risk curve in model (9) can be transformed into the following linear form

$$R^{-1}(x_1, x_2, \cdots, x_n; r) = x_1 \Phi_1^{-1}\Big(\alpha(r)\Big) + x_2 \Phi_2^{-1}\Big(\alpha(r)\Big) + \cdots + x_n \Phi_n^{-1}\Big(\alpha(r)\Big).$$

It follows from monotonicity of the uncertain measure that

$$x_1 \Phi_1^{-1}\Big(\alpha(r)\Big) + x_2 \Phi_2^{-1}\Big(\alpha(r)\Big) + \cdots + x_n \Phi_n^{-1}\Big(\alpha(r)\Big) \geq r_f - r.$$

When r degenerates to one specific number r_0, the risk curve becomes

$$\begin{aligned} R(x_1, x_2, \cdots, x_n; r_0) &= \mathcal{M}\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq r_0\} \\ &= \mathcal{M}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \leq r_f - r_0\} \end{aligned} \tag{11}$$

which is just the occurrence chance of one sensitive low return event (or say sensitive high loss event). If we use expression (11) as the risk measure, a portfolio is safe when

$$\mathcal{M}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \leq d_0\} \leq \alpha_0 \tag{12}$$

where d_0 is the sensitive low return value and α_0 the pre-set confidence level. It is seen that a safe portfolio judged by formula (12) may still be risky when judged by formula (8). However, a safe portfolio judged by formula (8) will still be safe when judged by formula (12). That is, an investor who adopt risk curve as the risk measure is very cautious and is concerned about every likely loss event.

7. Conclusions

Since the security markets are such complex markets that the randomness of the security returns are questioned by many scholars. In this paper we discuss the portfolio selection problem in which security returns are neither random nor fuzzy. Assuming that the security returns are uncertain variable, the paper introduces a risk curve and develops a mean-risk model. In addition, the crisp form of the model is provided.

Acknowledgements

This work was supported by National Natural Science Foundation of China Grant No.70871011.

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