Reliable Dynamic Analysis of an Uncertain Shear Beam

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Abstract: In structural engineering, shear beams and their dynamic behavior play an important role in modeling, analysis and design of various types of structures subjected to a system of dynamic loads such as wind or earthquake excitations. However, in current procedures of dynamic analysis of shear beams, the presence of uncertainty in the system's mechanical properties and/or applied forces is not considered. In this work, a new method is developed for dynamic modal analysis of continuous uncertain shear beams subjected to uncertain external loads. First, an interval formulation is used to quantify the uncertainty present in the system's mechanical characteristics and/or magnitude of dynamic forces. Then, having the interval parameters, the bounds on modal responses of the continuous system are obtained leading to determination of the upper-bounds of total response that may be used for design purposes. An example problem that illustrates the behavior of the method and a comparison with Monte-Carlo simulations is presented. Using this new method, it has shown that obtaining the bounds on the dynamic response of a shear beam does not require an iterative procedure such as Monte-Carlo simulations.

1. Introduction

In reliable design, the performance of the system must be guaranteed over its lifetime. Moreover, dynamic analysis is a fundamental procedure for designing reliable systems that are subjected to dynamic forces. Shear beam models and their dynamic behavior play an important role in modeling, analysis and design of various types of structures. This model takes into account shear deformation and rotational inertia effects. The study of the shear beam subjected to dynamic loads such as earthquakes and wind is significant due to the following reasons:

1. The dynamic properties of the system are sufficiently simple that it serves as a convenient representation for illustrating many of the fundamental features of earthquake response of linear structures.

2. It introduces the dynamic analysis of continuous systems with significant shear effects.

3. The uniform Shear Beam serves as an approximate model for earthquake and wind response of a very important class of structures: moderately tall and regular buildings.

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In current procedures for dynamic analysis of shear beams, the existence of uncertainty in either mechanical properties of the system or the characteristics of forcing function is not routinely considered. These uncertainties can be attributed to physical imperfections, modeling inaccuracies and system complexities. In design process, uncertainty is accounted for by a combination of load amplification and strength reduction factors that are based on probabilistic models of historic data. However, consideration of the effects of uncertainty has been removed from current dynamic analysis of shear beams.

In this work, a new method is developed to perform dynamic analysis of a continuous shear beam subjected to a system of dynamic load with uncertainty present in the system's mechanical properties as well as in the magnitude of dynamic loads. An interval formulation is used to represent the presence of uncertainty. Using interval operations, the upper bounds of system's response are obtained which can be used for reliable design purposes. It is shown that this method can achieve the bounds on dynamic response without Monte-Carlo simulation procedure.

2. Shear Beam Theory-Definition and Historical Background

The shear beam theory was developed by Ukrainian/Russian-born scientist Stephen Timoshenko in the beginning of the 20th century (Timoshenko and Young 1951). The model takes into account shear deformation and rotational inertia effects, making it suitable for describing the behavior of short beams, sandwich composite beams or beams subject to high-frequency excitation when the wavelength approaches the thickness of the beam.

Physically, taking into account the added shear mechanisms of deformation effectively lowers the stiffness and natural frequency of the beam more noticeable for higher frequencies as the wavelength becomes shorter, and thus, the distance between opposing shear forces decreases. If the shear modulus of the beam material approaches infinity and thus, the beam becomes rigid in shear, and also, if rotational inertia effects are neglected, Timoshenko beam theory converges towards ordinary Euler-Bernoulli beam theory.

3. Deterministic Dynamic Modal Analysis

Considering a linear elastic constitutive model of a shear beam subjected to a suddenly applied distributed load (Figure 1):

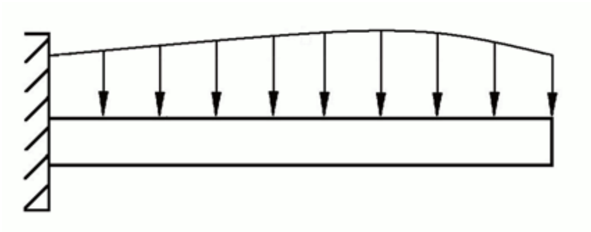


Figure 1. A general shear beam

The shear stress function in spatial and temporal domains is defined as:

$$\tau(x,t) = G\frac{\partial u(x,t)}{\partial x} \tag{1}$$

where, τ is the shear stress, G is shear modulus, u is the transverse displacement, x is longitudinal space variable and t is time variable.

Subsequently, the shear force function in spatial and temporal domains is:

$$V(x,t) = GA_s\frac{\partial u(x,t)}{\partial x} \tag{2}$$

where, $V(x,t)$ is the shear force, and $(A_s = k.A)$ is the shear area calculated as multiplication of shear form factor and the cross-sectional area.

Considering an infinitesimal segment of the shear beam, (Figure 2), the equilibrium in the transverse direction is:

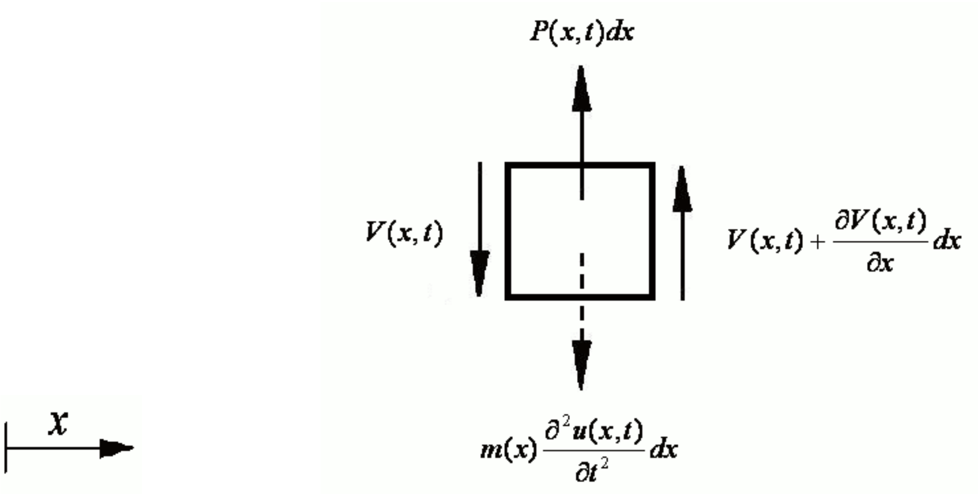


Figure 2. An infinitesimal segment of the shear beam

$$V(x,t) + \frac{\partial V(x,t)}{\partial x}dx - V(x,t) + P(x,t)dx - m(x)\frac{\partial^2 u(x,t)}{\partial t^2}dx = 0 \tag{3}$$

where, $P(x,t)$ is the applied external dynamic load defined in spatial and temporal domains, and $m(x)$ is the function of mass per unit length distribution defined in spatial domain.

Substituting Eq. 3 in Eq. 2, the partial differential equation of motion is:

$$\frac{\partial}{\partial x}\left(GA_s \frac{\partial u(x,t)}{\partial x} \right) + P(x,t) = m(x)\frac{\partial^2 u(x,t)}{\partial t^2} \tag{4}$$

which is a general partial differential equation in spatial and temporal domains for generic loadings, mass distribution and boundary conditions.

2.1. SOLUTION

Considering a homogeneous prismatic cantilever shear beam subjected to a suddenly applied uniformly distributed load, (Figure 3),

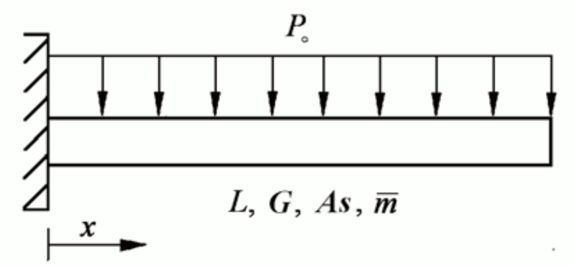


Figure 3. Cantilever shear beam with uniform dynamic load

the equation of motion, Eq. 4, is simplifies as:

$$\frac{\partial}{\partial x}\left(GA_s \frac{\partial u(x,t)}{\partial x} \right) + P_\circ = \overline{m}\frac{\partial^2 u(x,t)}{\partial t^2} \text{ a} \tag{5}$$

where, $P_\circ$ is the load ordinate and $\overline{m}$ is mass per unit length. The free vibration of the shear beam is:

$$\frac{\partial}{\partial x}\left(GA_s \frac{\partial u(x,t)}{\partial x} \right) = \overline{m}\frac{\partial^2 u(x,t)}{\partial t^2} \tag{6}$$

Assuming a harmonic solution of the form: $u(x,t) = \varphi(x)e^{i\omega t}$, in which $\varphi(x)$ is a spatial function and ω is the circular natural frequency, the linear eigenvalue problem becomes:

$$-\frac{\partial}{\partial x}\left(GA_s \frac{d\varphi(x)}{dx} \right) = \overline{m}\,\omega^2 \varphi(x) \tag{7}$$

Applying boundary conditions for the cantilever shear beam, ($\varphi(0) = \varphi'(L) = 0$), the solution for natural circular frequencies and corresponding mass-orthonormalized eige functions (mode shapes) are:

$$\omega_n = (2n-1)\frac{\pi}{2}\sqrt{\frac{GA_s}{\overline{m}L^2}} \tag{8}$$

$$\hat{\varphi}_n(x) = \sqrt{\frac{2}{\overline{m}L}}\sin\left((2n-1)\frac{\pi x}{2L}\right) \tag{9}$$

where, n is the mode number.

The solution for the forced vibration may be expressed as:

$$u(x,t) = \sum_{n=1}^{\infty} y_n(t)\hat{\varphi}_n(x) \tag{10}$$

where, $y_n(t)$ are the modal coordinates.

Substituting Eq. 10 in the governing equation, Eq. 4, premultiplying by $\hat{\varphi}_n(x)$, integrating over the domain, decoupling and adding modal damping ratio (ζ_n), the modal equation becomes:

$$\ddot{y}_n(t) + 2\zeta_n\omega_n\dot{y}_n(t) + \omega_n^2 y_n(t) = \int_0^L \hat{\varphi}_n(x)P_\circ dx \tag{11}$$

or:

$$\ddot{y}_n(t) + 2\zeta_n\omega_n\dot{y}_n(t) + \omega_n^2 y_n(t) = \frac{2P_o}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos\left(\frac{(2n-1)\pi L}{2L}\right)\right) \tag{12}$$

Defining modal participation factor as:

$$\Gamma_n = \frac{2P_o}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos\left(\frac{(2n-1)\pi L}{2L}\right)\right) \tag{13}$$

Also, defining a scaled generalized modal coordinate:

$$d_n(t) = \frac{y_n(t)}{\Gamma_n} \tag{14}$$

Eq. (14) is rewritten in terms of the scaled modal coordinate, $d_n(t)$, as:

$$\ddot{d}_n(t) + 2\zeta_n \omega_n \dot{d}_n(t) + \omega_n^2 d_n(t) = 1 \tag{15}$$

For each decoupled generalized modal equation, Eq. 15, the maximum modal coordinate is obtained from the response spectrum (maximum ratio of dynamic to static response) for modal frequency and assumed modal damping ratio (Figure 4).

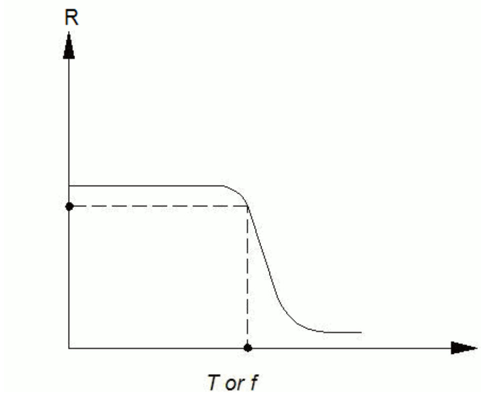


Figure 4. A generic response spectrum.

Then, the maximum modal displacement response is obtained as the multiplication of the maximum modal coordinate, modal participation factor, and mode shape as:

$$u_{n,\max} = (d_{n,\max})(\Gamma_n)(\hat{\varphi}_n(x)) \tag{16}$$

or:

$$u_{n,\max} = (d_{n,\max}) \frac{8\sqrt{2}P_o}{\overline{m}\sqrt{(2n-1)\pi}} \sin^3\left((2n-1)\frac{\pi x}{4L} \right) \cos\left((2n-1)\frac{\pi x}{4L} \right) \tag{17}$$

Finally, the total displacement response is obtained using superposition of modal maxima. The superposition can be performed by considering Square Root of Sum of Squares (SRSS) of modal maxima as (Rosenblueth 1962):

$$u_{\max} = \sqrt{\sum_{n=1}^{\infty} u^2{}_{n,\max}} \tag{18}$$

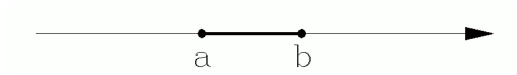
For practical purposes, the infinite series must be truncated. For systems with different patterns of load and boundary conditions, the same procedures can be used.

4. Interval Variables

The concept of interval numbers has been originally applied in the error analysis associated with digital computing. Quantification of the uncertainties introduced by truncation of real numbers in numerical methods was the primary application of interval methods (Moore 1966).

A real interval is a closed set defined by extreme values as (Figure 5):

$$\tilde{Z} = [z^l, z^u] = \{z \in \Re \mid z^l \le z \le z^u\} \tag{19}$$



$$\tilde{x} = [a, b]$$

Figure 5. An interval variable.

One interpretation of an interval number is a random variable whose probability density function is unknown but non-zero only in the range of interval. Another interpretation of an interval number includes intervals of confidence for α -cuts of fuzzy sets. The interval representation transforms the point values in the deterministic system to inclusive set values in the system with bounded uncertainty. In this work, the symbol (~) represents an interval quantity.

5. Interval Dynamic Analysis

The partial differential equation of motion for a shear beam subjected to a with interval uncertainty in the shear modulus and magnitude of load is:

$$\frac{\partial}{\partial x}\left(\tilde{G}A_s \frac{\partial u(x,t)}{\partial x}\right) + \tilde{P}_o(x,t) = m(x)\frac{\partial^2 u}{\partial t^2} \tag{20}$$

where, $\tilde{G} = [G_l, G_u]$ and $\tilde{P}_o = [P_o^l, P_o^u]$. This equation is a general partial differential equation in spatial and temporal domains for generic interval loadings, mass distribution and boundary conditions.

Specializing the problem as a cantilever beam with uniform load, and solving this partial differential equation, the interval eigenvalue problem is:

$$-\frac{\partial}{\partial x}\left(\tilde{G}A_s \frac{d\varphi(x)}{dx}\right) = \overline{m}\,\tilde{\omega}_n \varphi(x) \tag{21}$$

Applying boundary conditions, the solution for natural circular frequencies and corresponding mode shapes are:

$$\tilde{\omega}_n = (2n-1)\frac{\pi}{2}\sqrt{\frac{\tilde{G}A_s}{\overline{m}L^2}} \tag{22}$$

$$\hat{\varphi}_n(x) = A_n \sin\left((2n-1)\frac{\pi \cdot x}{2L}\right) \tag{23}$$

Eq. 23 can be rewritten as:

$$\tilde{\omega}_n = (2n-1)\frac{\pi}{2}\left[\sqrt{\tilde{G}^L}, \sqrt{\tilde{G}^U}\right] \cdot \sqrt{\frac{A_s}{\overline{m}L^2}} \tag{24}$$

This shows that the lower bound of $\tilde{G}$, yields the lower bound of natural circular frequency and similarly, the upper bound of $\tilde{G}$ yields the upper bound of natural circular frequency. This leads to an evident realization of monotonic behavior of natural circular frequencies due to variation in stiffness in continuous dynamic systems.

In discrete systems, because of the complexity of the eigenvalue problem, this realization is not straightforward. Modares and Mullen (2004) proved this monotonic behavior of natural frequencies in discrete systems using monotonicity of eigenvalues for symmetric matrices subjected to non-negative definite perturbations.

The interval modal coordinate is determined using the excitation response spectrum evaluated for the corresponding interval of natural circular frequency and assumed modal damping ratio (Figure 6).

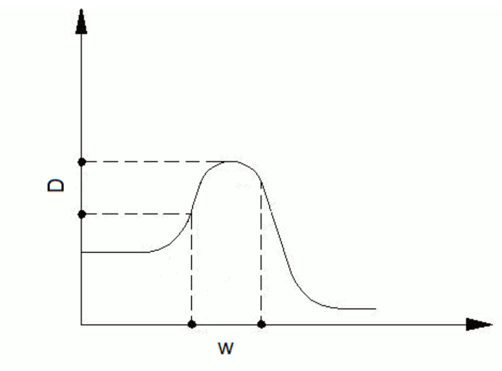


Figure 6. Determination of $\tilde{d}_n$ corresponding to a $\tilde{\omega}_n$ for a generic response spectrum

Having the interval modal coordinate, the maximum (upper-bound) modal coordinate $d_{n,\max}$ is:

$$d_{n,\max} = \max(\tilde{d}_n) \tag{25}$$

The interval modal participation factor is:

$$\tilde{\Gamma}_n = \frac{2\tilde{P}_o}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos\left(\frac{(2n-1)\pi L}{2L}\right)\right) \tag{26}$$

Therefore, the maximum participation factor is:

$$\Gamma_{n,\max} = \max(\tilde{\Gamma}_n) = \frac{P_o^{\,u}}{\sqrt{\overline{m}}(2n-1)\pi}\left(1-\cos\left(\frac{(2n-1)\pi L}{2L}\right)\right) \tag{27}$$

Then, the maximum modal displacement response is obtained as the multiplication of maximum modal coordinate, maximum modal participation factor and mode shape as:

$$u_{n,\max} = (d_{n,\max})(\Gamma_{n,\max})(\hat{\varphi}_n(x)) \tag{28}$$

or:

$$u_{n,\max} = (d_{n,\max})\frac{8\sqrt{2}P_o^{\,u}}{\overline{m}\sqrt{(2n-1)\pi}}\sin^3\left((2n-1)\frac{\pi x}{4L}\right)\cos\left((2n-1)\frac{\pi x}{4L}\right) \tag{29}$$

Finally, the total displacement response is obtained using superposition of modal maxima. The superposition can be performed by considering Square Root of Sum of Squares (SRSS) of modal maxima as (Rosenblueth 1962):

$$u_{\max} = \sqrt{\sum_{n=1}^{\infty} u^2{}_{n,\max}} \tag{30}$$

in which, the upper bound of the response in the presence of uncertainty in both stiffness and load is obtained.

6. Numerical Example

The example obtains the bounds on maximum dynamic displacement for a continuous shear cantilever beam with interval uncertainty in the shear modulus and magnitude of load using present method and compares the results with Monte-Carlo simulations (Figure 7).

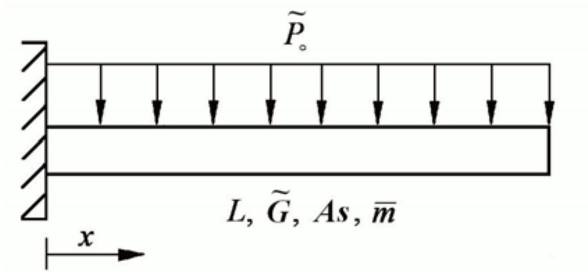


Figure 7. Cantilever shear beam with uncertainty in shear modulus and magnitude of uniform dynamic load

The beam's length is $L = 10\,ft$, mass is $\overline{m} = 132 lb/g$ per foot, the moment of inertia is $I = 1530\ in^4$, assumed modal damping ratio is $\zeta = 1\%$, shear area is $A_s = 25.23\ in^2$ and uncertain modulus of elasticity is $G = ([0.9,1.1])11200ksi$. The load's uncertain magnitude of load is $\widetilde{P}_\circ = [0.9,1.1]P_\circ$.

6.1 Solution

The problem is solved using the present method and the bounds on dynamic response of the beam are obtained. These results are compared with Monte-Carlo simulation solution using bounded uniformly distributed random variables in 1000 simulations. The solution for bounds on the first (fundamental) natural circular frequency is obtained and summarized in table I.

Table I. Bounds on Natural Circular Frequencies

	Lower Bound *Present Method*	Lower Bound *Monte-Carlo Simulation*	Upper Bound *Monte-Carlo Simulation*	Upper Bound *Present Method*
ω_1	39.12162	39.12733	43.24335	43.25451

The upper bounds of the maximum displacement response at the beam's free end for the first three modes are obtained and summarized in table II.

Table II. Upper bounds of free-end displacement response

	Upper Bound *Monte-Carlo Simulation*	Upper Bound *Present Method*
$\frac{u_{\max}}{P_{\circ}}$	0.350714×10^{-3}	0.350816×10^{-3}

The mode shapes for the first three modes are shown in figure 8.

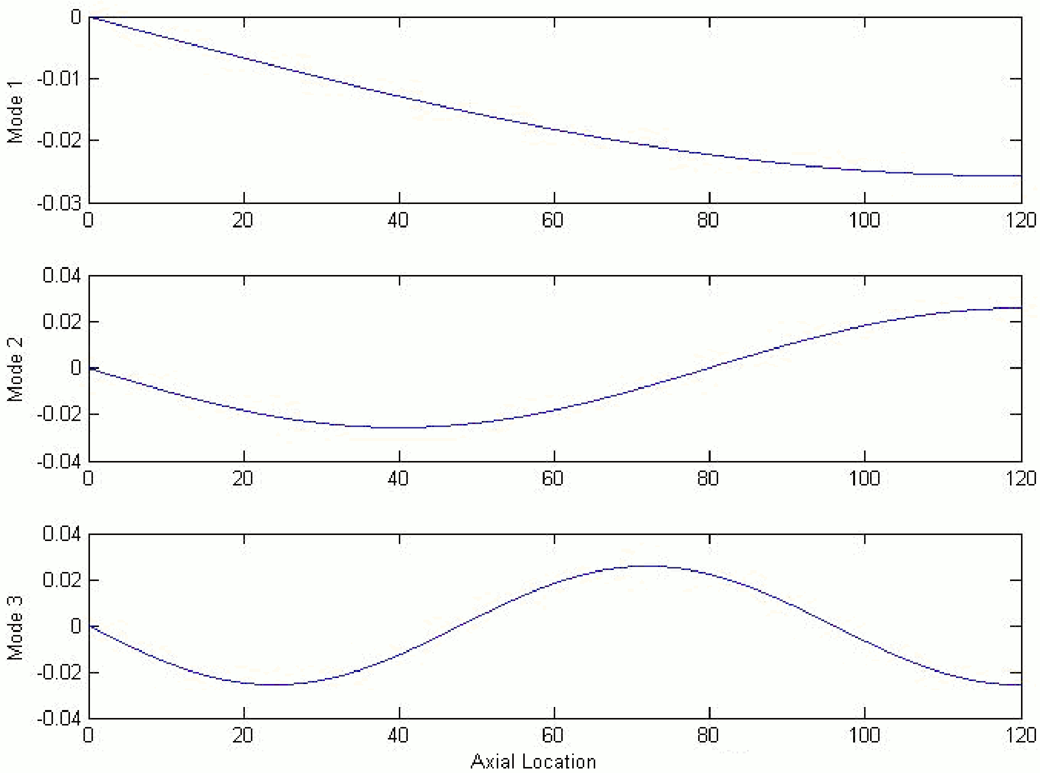


Figure 8. Beam deflection for the first three-mode response for example problem

The first three-mode beam response is shown in figure 9. The continuous line depicts the bounds obtained by the present method and the dashed-lines depict the bounds obtained by Monte-Carlo simulation solution.

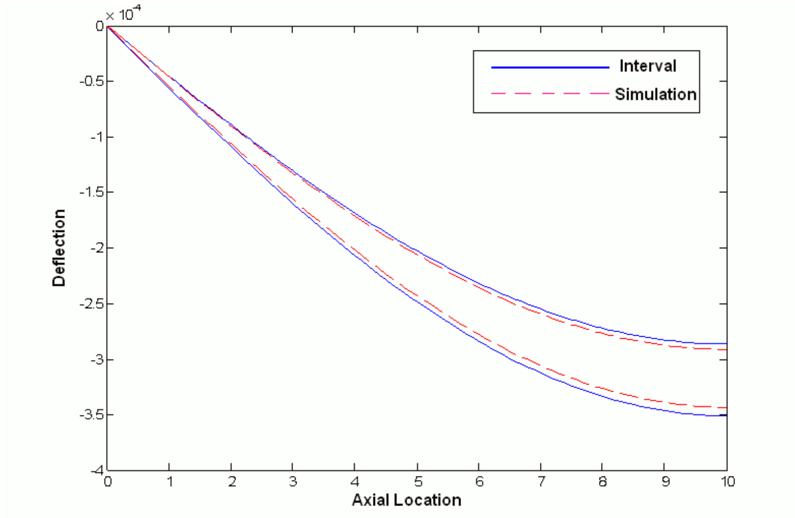


Figure 9. Beam deflection for the first three-mode response for example problem

6.2 Sensitivity Analysis

The beam in the numerical example is used for a sensitivity analysis to show the robustness of the proposed method. The shear modulus of elasticity is varied as $G = 1 + n \times [-0.1, 0.1] \times G_{Central}$ in which, $n = 1, 2, ...5$, and $G_{Central} = 11200ksi$. The results for free end response are shown in Figure 10.

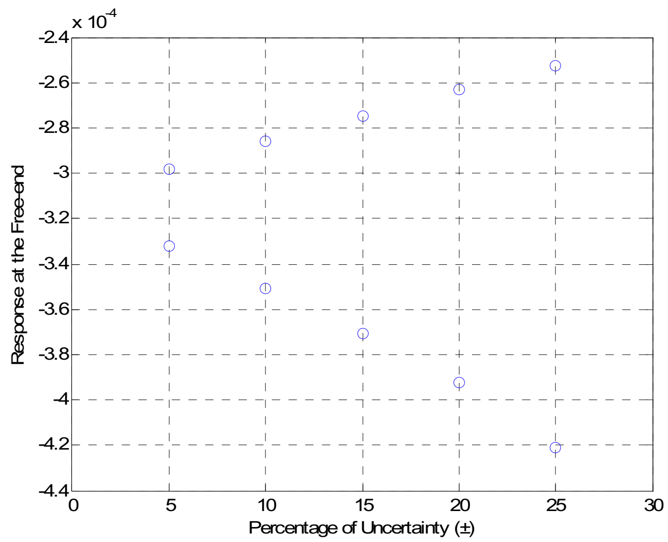


Figure 10. Sensitivity Analysis Results

The results of sensitivity analysis show that the growth in width of interval of dynamic response is linear. Also, it can be inferred that in the case of increasing uncertainty, the proposed method is robust.

The results show that using the proposed method, the system's physics is preserved and also, the obtained sharp solutions are upper-bounds to solutions obtained by methods that produce inner-bound results such as Monte-Carlo simulation. Moreover, the proposed method is computationally feasible because of its non-iterative process.

7. Conclusions

A new method for dynamic analysis of shear beam with uncertainty in the mechanical characteristics of the system as well as the magnitude of the external applied load is developed.

This computationally efficient method shows that implementation of interval analysis in a continuous dynamic system yields exact and robust results and preserves the problem's physics.

The sharpness of results may be attributed to completeness of the closed-form solution in continuous dynamic systems.

The results show that obtaining bounds does not require expensive stochastic procedures such as Monte-Carlo simulations.

The simplicity of the proposed method makes it attractive to introduce uncertainty in analysis of continuous dynamic systems.

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