Interval Estimates of Structural Reliability

Hao Zhang
School of Civil Engineering, University of Sydney, NSW 2006, Australia

Robert L. Mullen
Department of Civil Engineering, Case Western Reserve University, Cleveland, OH 44106, USA

Rafi L. Muhanna
School of Civil and Environmental Engineering, Georgia Institute of Technology, Savannah, GA 31407, USA

Abstract. This paper demonstrates the feasibility of computing interval estimates of structural reliability when distribution functions of basic random variables have uncertain parameters. The incomplete knowledge regarding the distribution parameters is modeled by interval variables. Reliability analysis needs to consider families of distributions whose parameters are within the intervals, leading to an interval estimate of the failure probability. To estimate the interval failure probability, an interval Monte Carlo method has been proposed which combines simulation process with the interval finite element analysis. The authors' previously developed interval finite element method is utilized to model the ranges of structural responses. The developed method propagates the epistemic uncertainty and aleatory uncertainty separately through the reliability assessment. A planar steel frame example is presented to demonstrate the proposed method.

Keywords: Epistemic uncertainty, Interval analysis, Interval finite element, Monte Carlo simulation, Parameter uncertainty, Probability box, Structural reliability

1. Introduction

A major step in structural reliability analysis is the modeling and quantification of various sources of uncertainty. It is common in engineering practice to distinguish between "aleatory" uncertainty and "epistemic" uncertainty. In contrast to aleatory uncertainty which is due to intrinsic variability of physical quantities, epistemic uncertainty arises from imperfect modeling, simplification and limited availability of database (Melchers, 1999). Statistical uncertainty is an important source of epistemic uncertainty. The probability distribution for modeling a random phenomenon is generally not precisely known. The statistical parameters (e.g., mean and standard deviation) are usually estimated by statistical inference from a limited sample of observational data and a point estimator is used to approximate the "true" parameter. Statistical independent relationship between random variables is often assumed without validation with sufficient data. Thus the distribution is itself subject to some uncertainty. This additional source of epistemic uncertainty can play an important role and should be considered in reliability analysis and risk assessment (Ellingwood, 2001).

A very popular method of considering epistemic uncertainty is the Bayesian approach. The epistemic uncertainty is modeled to be (Bayesian) random variables (Der Kiureghian, 2008). Subjective

4th International Workshop on Reliable Engineering Computing (REC 2010)
Edited by Michael Beer, Rafi L. Muhanna and Robert L. Mullen

ISBN: 978-981-08-5118-7. *Published by* Research Publishing Services.
doi:10.3850/978-981-08-5118-7_053

judgment is incorporated with any observational data by the Bayesian updating formula. The estimate of the Bayesian random variables can be improved when more data become available. Before receiving additional data, however, the Bayesian approach remains a subjective representation of uncertainty. More recently a number of frameworks for imprecise probability theories have been proposed, including fuzzy set theory, imprecise probability, fuzzy randomness, interval approach and others. A state-of-the-art review on the non-traditional uncertainty models is provided in (Möller and Beer, 2008).

This paper considers the epistemic uncertainties in specifying the probabilistic models of loads and resistances of structures. The incomplete knowledge regarding the distribution parameters and statistical dependency (coefficient of correlation) between basic random variables are modeled by intervals. Reliability assessment needs to consider families of distributions whose parameters are within the intervals. One practical way to describe the ensemble of distributions is to specify its lower and upper bounds. The mathematical frameworks using this methodology include Dempster-Shafer evidence theory (Dempster, 1967), random set theory (Kendall, 1974) and probability boxes (Ferson et al., 2003). The computation procedures are typically a combination of interval analysis and Cartesian products that depend on the assumption about the dependency between variables (Williamson, 1989). Lower and upper bounds on the limit state probability are computed. These methods have been applied to model epistemic uncertainty in reliability and risk analyses (Penmetsa and Grandhi, 2002; Tonon et al., 2006; Hajagos, 2007; Oberguggenberger and Fellin, 2008; Qiu et al., 2008, among others).

Despite the research progress, the computing effort, especially when Cartesian product method is used, is a barrier to the practical application of non-traditional uncertainty models. The issue of computational cost becomes more serious when the reliability analysis is finite element (FE) based, i.e., the structural responses are obtained through FE analyses. To overcome the computational issue of the Cartesian product method, an interval Monte Carlo method is proposed to propagate interval parameters through FE-based reliability assessment.

2. Interval Monte Carlo Method

2.1. Basic formula

The basic reliability problem is to estimate the probability that an engineered system will adequately perform its intended function over a given period of time. The probability of failure p_f is given by

$$p_f = P(G(\mathbf{X}) \leq 0) = \int \cdots \int_{G(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \tag{1}$$

Here $\mathbf{X} = (X_1, \cdots, X_n)^{\mathrm{T}}$ is the n-dimensional vector of the basic random variables representing uncertain quantities such as applied loads, material strength and stiffness. $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for $\mathbf{X}$. $G(\mathbf{X})$ is the limit state function and $G(\mathbf{X}) \leq 0$ defines the failure state. The integration in Eq. (1) can be approximated using the methods such as the simulation methods and the first-order second-moment method (Melchers, 1999). In the case of the Monte

Carlo method, Eq. (1) is approximated by

$$p_f \approx \frac{1}{N}\sum_{j=1}^{N} I[G(\hat{\mathbf{x}}_j) \leq 0] \tag{2}$$

where N is the total number of simulations conducted, $\hat{\mathbf{x}}_j$ represents the jth randomly simulated vector of basic variables, and $I[\quad]$ is the indicator function, having the value 1 if $[\quad]$ is "true" and the value 0 if $[\quad]$ is "false".

Suppose some statistical parameters, θ, of $f_{\mathbf{X}}(\mathbf{x})$ are uncertain and vary in intervals. Let Θ denote the intervals, and θ is a generic (arbitrary) element $\theta \in \Theta$. Under this assumption one needs to consider families of distributions whose parameters are in the intervals. A visualization of all possible distributions with $\theta \in \Theta$ can be obtained by means of upper and lower distribution functions. Let $F(x)$ denote the cumulative distribution function (CDF) for the random variable X. For every x, an interval $[\underline{F}(x),\ \overline{F}(x)]$ generally can be readily found to bound the possible values of $F(x)$, i.e., $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$, for $\theta \in \Theta$. Such a pair of two CDFs $\underline{F}(x)$ and $\overline{F}(x)$ construct a so-called *probability box* or *probability bounds* (Ferson et al., 2003).

Consider Eq. (2). When θ vary in intervals, the randomly simulated basic variables $\hat{\mathbf{x}}_j$ also vary in intervals accordingly. The limit state function $G(\hat{\mathbf{x}}_j)$ becomes a function of θ as well, i.e., $G(\hat{\mathbf{x}}_j,\ \theta)$. If the minimum and maximum values of $G(\hat{\mathbf{x}}_j,\ \theta)$ can be determined

$$\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq G(\hat{\mathbf{x}}_j,\ \theta) \leq \text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)), \quad \text{for } \theta \in \Theta \tag{3}$$

then

$$I[\text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0] \leq I[G(\hat{\mathbf{x}}_j,\ \theta) \leq 0] \leq I[\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0]. \tag{4}$$

Applying Eq. (4) in (2) gives

$$\frac{1}{N}\sum_{j=1}^{N} I[\text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0] \leq \frac{1}{N}\sum_{j=1}^{N} I[G(\hat{\mathbf{x}}_j,\ \theta) \leq 0] \leq \frac{1}{N}\sum_{j=1}^{N} I[\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0]. \tag{5}$$

Therefore, Eq. (5) provides an interval estimate for p_f

$$\begin{aligned} \underline{p}_f &\approx \frac{1}{N}\sum_{j=1}^{N} I[\text{Max}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0], \\ \overline{p}_f &\approx \frac{1}{N}\sum_{j=1}^{N} I[\text{Min}\ (G(\hat{\mathbf{x}}_j,\ \theta)) \leq 0], \quad \text{for } \theta \in \Theta. \end{aligned} \tag{6}$$

2.2. Computational aspects

The first step in the implementation of the interval Monte Carlo method is the generation of intervals in accordance with the prescribed probability boxes. The inverse transform method (Ang and Tang, 1975) can be utilized for this purpose. Consider a random variable X with CDF $F(x)$. If $(u_1, u_2, \cdots, u_m)$ is a set of values from the standard uniform variate, then the set of values

$$x_i = F_X^{-1}(u_i);\ i = 1, 2, \cdots, m \tag{7}$$

will have the desired CDF $F(x)$. The inverse transform method can be extended to perform random sampling from a probability box. Suppose an imprecise CDF $F(x)$ bounded by $\overline{F}(x)$ and $\underline{F}(x)$, as shown in Fig. (1). For each u_i in Eq. (7), two random numbers are generated

$$\underline{x}_i = \overline{F}^{-1}(u_i),\ \overline{x}_i = \underline{F}^{-1}(u_i). \tag{8}$$

The method is graphically demonstrated in Fig. 1 for one-dimensional case. Such a pair of $\underline{x}_i$ and $\overline{x}_i$ form an interval $[\underline{x}_i,\ \overline{x}_i]$ which contains all possible simulated numbers from the ensemble of distributions for a particular u_i.

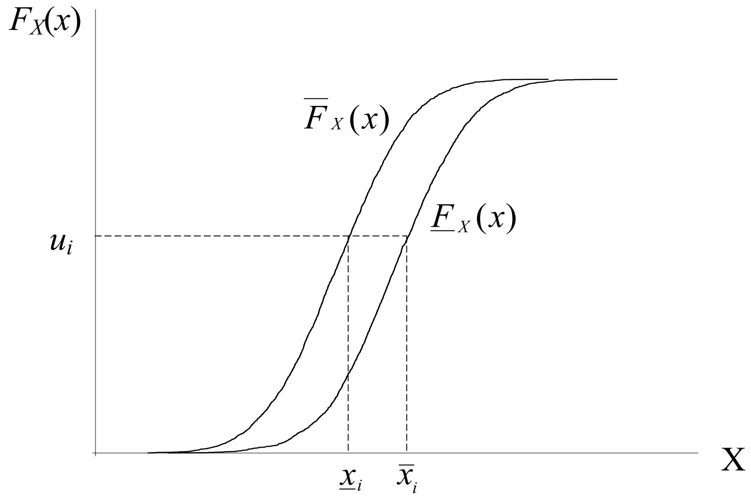

Figure 1. Generation of random intervals from probability box.

2.3. Computing the ranges of structure responses – interval FEM

The computational effort of the interval Monte Carlo method is contingent on the efficiency of computing the range (max. and min.) of structural responses through FE analyses when the simulated basic variables vary in intervals. This task can be carried out by using the interval FEM. A variety of solution techniques have been proposed for interval FEM, including the combinatorial method (Rao and Berke, 1997; Ganzerli and Pantelides, 1999), perturbation method (Qiu and Elishakoff, 1998; McWilliam, 2000), sensitivity analysis method (Pownuk, 2004), optimization method (Koyluoglu et al., 1995; Möller and Beer, 2004), and interval analysis method (Mullen and Muhanna, 1999; Muhanna and Mullen, 2001; Dessombz et al., 2001). In this paper, the interval FE analysis is formulated as an interval analysis problem. The interval analysis and interval FE formulation is briefly introduced below. Further details are provided in the authors' previous work (Muhanna et al., 2005; Zhang, 2005; Muhanna et al., 2007).

Interval analysis concerns the numerical computations involving interval numbers. The four elementary operations of real arithmetic, namely addition (+), subtraction (−), multiplication (×) and division (÷) can be extended to intervals. Operations $\circ \in \{+, -, \times, \div\}$ over interval numbers $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined by the general rule (Moore, 1966; Neumaier, 1990)

$$\boldsymbol{x} \circ \boldsymbol{y} = [\min\, (x \circ y),\ \max\, (x \circ y)] \quad \text{for } \circ \in \{+, -, \times, \div\} \tag{9}$$

in which x and y denote generic elements $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$. Software and hardware support for interval computation are available (e.g. (Sun microsystems, 2002; Knüppel, 1994)).

For a real-valued function $f(x_1, \ldots, x_n)$, the *interval extension* of $f(\ )$ is obtained by replacing each real variable x_i by an interval variable $\boldsymbol{x}_i$ and each real operation by its corresponding interval arithmetic operation. From the fundamental property of *inclusion isotonicity* (Moore, 1966), the range of the function $f(x_1, \ldots, x_n)$ can be rigorously bounded by its interval extension function

$$\{f(x_1, \ldots, x_n) \mid x_1 \in \boldsymbol{x}_1, \ldots, x_n \in \boldsymbol{x}_n\} \subseteq f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n). \tag{10}$$

Eq. (10) indicates that $f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ contains the range of $f(x_1, \ldots, x_n)$ for all $x_i \in \boldsymbol{x}_i$.

A natural idea to implement interval FEM is to apply the interval extension to the deterministic FE formulation. Unfortunately, such a naïve use of interval analysis in FEM yields meaningless and overly wide results (Muhanna and Mullen, 2001; Dessombz et al., 2001). The reason is that in interval arithmetic each occurrence of an interval variable is treated as a *different, independent* variable. It is critical to the formulation of the interval FEM to identify the dependence between the interval variables and prevent the widening of results. In this paper, an element-by-element (EBE) technique is utilized for element assembly (Muhanna and Mullen, 2001; Zhang, 2005). The elements are detached so that there are no connections between elements, avoiding element coupling. The penalty method is then employed to impose constraints to recover the connections between elements, and to ensure the compatibility of displacements. The system equation in the interval FEM takes the following form

$$(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p} \tag{11}$$

where $\boldsymbol{K}$ is the interval stiffness matrix, $\boldsymbol{u}$ is the interval displacement vector, $\boldsymbol{p}$ is the interval load vector, and Q is the deterministic penalty matrix. Eq. (11) can be transformed into a fixed point equation

$$R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0 + (I - R(\boldsymbol{K} + Q))\boldsymbol{u}^* = \boldsymbol{u}^* \tag{12}$$

in which R is a nonsingular preconditioning matrix, and u_0 is an approximate deterministic solution. It can be verified that $\boldsymbol{u} = \boldsymbol{u}^* + u_0$. From Eq. (12), interval fixed point iterations can be constructed (Neumaier, 1990; Rump, 1990)

$$\boldsymbol{Z} + \boldsymbol{C}\boldsymbol{u}^{*(l)} = \boldsymbol{u}^{*(l+1)} \tag{13}$$

with $\boldsymbol{Z} = R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0$, and the iterative matrix $\boldsymbol{C} = I - R(\boldsymbol{K} + Q)$. The iterations are terminated if

$$\boldsymbol{u}^{*(l+1)} \subseteq \boldsymbol{u}^{*(l)}. \tag{14}$$

Then $\boldsymbol{u}^{*(l+1)} + u_0$ guarantees to contain the exact solution set of Eq. (11). The original interval fixed point iteration implicitly assumes that the coefficients of $\boldsymbol{K}$ vary independently between their bounds. This assumption is not valid for the system equations that arise in the interval FEM. Special formulation has to be developed to remove coefficient-dependence in the algorithm. By using the EBE technique, it is possible to decompose the interval stiffness matrix $\boldsymbol{K}$ into two parts

$$\boldsymbol{K} = S\boldsymbol{D} \tag{15}$$

in which S is a deterministic matrix and $\boldsymbol{D}$ is an interval diagonal matrix whose diagonal entries are the interval variables associated with each element (e.g., interval modulus of elasticity). The first term $\boldsymbol{Z}$ in Eq. (13) can then be reintroduced as

$$\begin{aligned} \boldsymbol{Z} &= R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0 \\ &= R\boldsymbol{p} - RQu_0 - RS\boldsymbol{D}u_0 \\ &= R\boldsymbol{p} - RQu_0 - RSM\boldsymbol{\delta}. \end{aligned} \tag{16}$$

It must be noted that in Eq. (16) $\boldsymbol{D}u_0$ is introduced as $M\boldsymbol{\delta}$ in which M is a deterministic matrix and $\boldsymbol{\delta}$ is an interval vector. The components of $\boldsymbol{\delta}$ are the diagonal entries of $\boldsymbol{D}$ with the difference that every interval variable occurs only once in $\boldsymbol{\delta}$. This treatment eliminates the coefficient-dependence in $\boldsymbol{Z}$, which is critical for obtaining a tight bound.

The interval fixed point iteration converges if and only if $\rho(|\boldsymbol{C}|) < 1$ (Rohn and Rex, 1998), where $\rho(|\boldsymbol{C}|)$ is the spectral radius of the absolute value of the iterative matrix $\boldsymbol{C}$. To achieve a small $\rho(|\boldsymbol{C}|)$, the choice $R = (Q + S)^{-1}$ is made such that

$$\begin{aligned} \boldsymbol{C} &= I - R(Q + S\boldsymbol{D}) \\ &= I - R(Q + S) - RS(\boldsymbol{D} - I)) \\ &= -RS(\boldsymbol{D} - I)). \end{aligned} \tag{17}$$

Numerical tests have shown that fast convergence (within 10 iterations) generally can be achieved by using the above modified iterative algorithm. The developed linear elastic interval FEM has been successfully applied to plane frame structures, as well as continuum mechanics problems (Muhanna et al., 2005; Zhang, 2005; Muhanna et al., 2007). The structural responses can be accurately and efficiently computed.

3. Example: plane frame

A two-bay two-story steel frame shown in Fig. 2 is considered. The frame geometry and member sizes are based on those of (Ziemian, 1990). In the figure the column is denoted as “C” and the beam as “B”. Subscripts indicate member number. The frame is subjected to lateral loads at each floor and vertical loads at the top. The Young’s modulus is 200 GPa. Linear elastic analyses were performed. Two cases are considered.

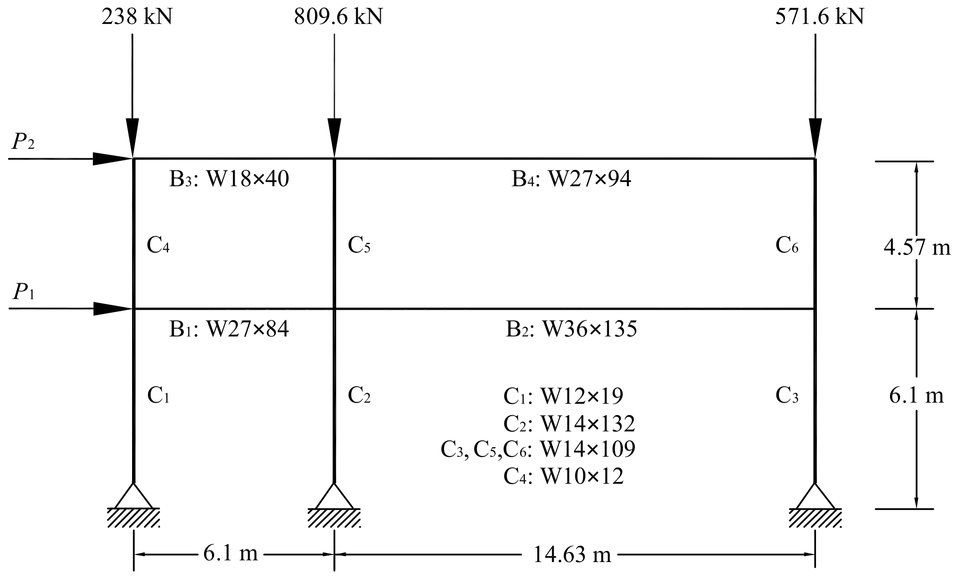

Figure 2. Two-bay two-story frame.

3.1. Case 1

The loads are considered as deterministic. The two lateral loads p_1 and p_2 are 33 kN and 14 kN, respectively. The sectional properties (moment of inertia I and cross-sectional area A) of each member are considered as random variables. Suppose only interval bound information is available for the means of the random variables. Table I summarizes the interval means and the standard deviations for the basic random variables. All random variables are assumed normals. For simplicity, it is assumed that the moment of inertia and cross-sectional area are statistically independent. Perfectly correlation is assumed between column-to-column, and between beam-to-beam. No correlation exists between column-to-beam.

The system limit state considered in Case 1 is the roof drift of $H/500$ (21 mm), where H is the height of the frame. The bounds of the cumulative frequency distribution of the roof drift were obtained after 10,000 interval Monte Carlo simulations, and is as shown in Fig. 3. The probability of failure is found to be between 0.11% and 1.29%. The width of the interval failure probability indicates the effect of the epistemic uncertainties on reliability assessment.

3.2. Case 2

In Case 2, the lateral loads p_1 and p_2 are identified as normal random variables with mean values of 33 kN and 14 kN, respectively, and COV (coefficient of variation) of 0.2 for both p_1 and p_2. The coefficient of correlation ρ between p_1 and p_2 is consider uncertain, and assumed to vary between 0.5 and 0.9. The vertical loads and member properties are assumed to be deterministic. Two drift limit states are considered: $H/450$ (23.7 mm) and $H/400$ (26.7 mm). The eigenvalue transformation method (Melchers, 1999) was used to transform the correlated p_1 and p_2 into two uncorrelated

Table I. Random sectional properties for the frame of Fig. 2 (Case 1).

Member	μ_I (cm^4)	σ_I (cm^4)	μ_A (cm^2)	σ_A (cm^2)
C_1	[5327.2, 5494.9]	270.55	[35.4, 36.5]	1.80
C_2	[62696.6, 64670.2]	3184.2	[246.4, 254.2]	12.5
C_4	[2204.6, 2274.0]	111.97	[22.5, 23.2]	1.14
C_3 C_5 C_6	[50813.0, 52412.4]	2580.6	[203.3, 209.7]	10.32
B_1	[116787.9, 120464.1]	5931.3	[157.5, 162.5]	8.0
B_2	[319629.9, 329691.1]	16233.0	[252.2, 260.1]	12.81
B_3	[25078.7, 25868.1]	1273.7	[75.0, 77.3]	3.81
B_4	[133998.7, 138216.6]	6805.4	[176.0, 181.5]	8.94

μ: mean; σ: standard deviation.

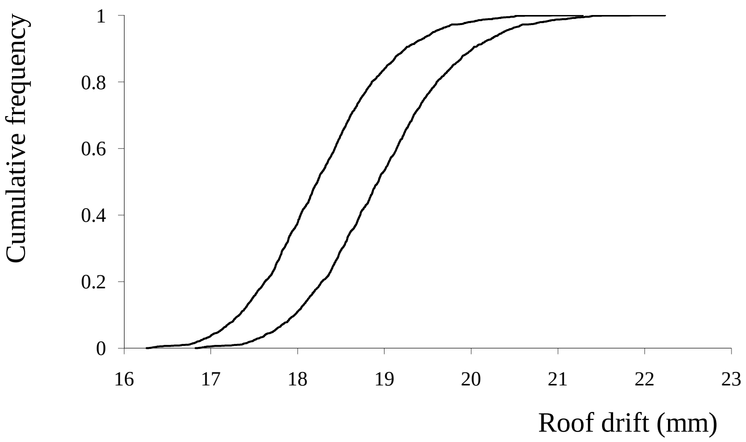

Figure 3. Bounds of the cumulative frequency distribution of the roof drift (Case 1).

normals. 100,000 interval Monte Carlo simulations were performed. The interval failure probability was found to be $[0.56\%, 1.9\%]$ for the drift criteria of $H/400$, and $[5.2\%, 10.4\%]$ for the criteria of $H/450$. It is evident that the coefficient of correlation between p_1 and p_2 has significant effect on the system reliability. Additional data for the loads should be collected if more confidence in the reliability estimate is needed.

4. Conclusion

An interval Monte Carlo method has been proposed for reliability assessment under parameter uncertainties represented by interval variables. The interval information of unknown parameters and the inherent uncertainties are propagated separately through reliability analysis. Interval FEM is utilized to model the ranges of structural responses accurately and efficiently. Interval estimates of failure probability are computed which can provide a statement of confidence in the results of the reliability estimate. A wide interval p_f implies that epistemic uncertainties are large, thus additional data should be collected. The developed method can also be used to study the sensitivities of failure probability with respect to changes in distribution parameters. A plane steel frame has been analyzed to illustrate the proposed method. Roof drift is considered as the system limit state. The means of the member sectional properties and the coefficient of correlation of the lateral loads are considered as intervals. Interval failure probabilities were computed. The results show that the structural reliability is mainly influenced by the uncertainties in the loads.

References

Ang, A. H.-S. and Tang, W. (1975). *Probability concepts in engineering planning and design, Vol.1-basic principles.* John Wiley.

Dempster, A. P. (1967). “Upper and lower probabilities induced by a multi-valued mapping.” *Ann. Mat. Stat.*, 38, 325–339.

Der Kiureghian, A. (2008). “Analysis of structural reliability under parameter uncertainties.” *Probab. Engrg. Mech.*, 23, 351–358.

Dessombz, O., Thouverez, F., Laîné, J.-P., and Jézéquel, L. (2001). “Analysis of mechanical systems using interval computations applied to finite elements methods.” *J. Sound. Vib.*, 238(5), 949–968.

Ellingwood, B. (2001). “Earthquake risk assessment of building structures.” *Reliability Engineering & System Safety*, 74(3), 251–262.

Ferson, S., Kreinovich, V., Ginzburg, L., Myers, D. S., and Sentz, K. (2003). “Constructing probability boxes and Dempster-Shafer structures.” *Report No. SAND2002-4015*, Sandia National Laboratories.

Ganzerli, S. and Pantelides, C. P. (1999). “Load and resistance convex models for optimum design.” *Struct. Optim.*, 17, 259–268.

Hajagos, J. G. (2007). “Interval Monte Carlo as an alternative to second-order sampling for estimating ecological risk.” *Reliable Computing*, 13, 71–81.

Kendall, D. G. (1974). “Foundations of a theory of random sets.” *Stochastic Geometry*, E. Harding and D. Kendall, eds., New York. Wiley, 322–376.

Knüppel, O. (1994). “Profil/bias-a fast interval library.” *Computing*, 53, 277–288.

Koyluoglu, U., Cakmak, S., Ahmet, N., and Soren, R. K. (1995). “Interval algebra to deal with pattern loading and structural uncertainty.” *J. Engrg. Mech., ASCE*, 121(11), 1149–1157.

McWilliam, S. (2000). “Anti-optimisation of uncertain structures using interval analysis.” *Comput. Struct.*, 79, 421–430.

Melchers, R. E. (1999). *Structural reliability analysis and prediction.* John Wiley & Sons, West Sussex, England, 2 edition.

Möller, B. and Beer, M. (2004). *Fuzzy-randomness-uncertainty in civil engineering and computational mechanics.* Springer-Verlag.

Möller, B. and Beer, M. (2008). “Engineering computation under uncertainty– capabilities of non-traditional models.” *Comput. Struct.*, 86(10), 1024–1041.

Moore, R. E. (1966). *Interval Analysis.* Prentice-Hall, Inc., Englewood Cliffs, N. J.

Muhanna, R. L. and Mullen, R. L. (2001). “Uncertainty in mechanics problems-interval-based approach.” *J. Engrg. Mech., ASCE*, 127(6), 557–566.

Muhanna, R. L., Mullen, R. L., and Zhang, H. (2005). “Penalty-based solution for the interval finite element methods.” *J. Struct. Engrg., ASCE*, 131(10), 1102–1111.

Muhanna, R. L., Zhang, H., and Mullen, R. L. (2007). “Combined axial and bending stiffness in interval finite-element methods.” *J. Struct. Engrg., ASCE*, 133(12), 1700–1709.

Mullen, R. L. and Muhanna, R. L. (1999). "Bounds of structural response for all possible loadings." *J. Struct. Engrg., ASCE*, 125(1), 98–106.

Neumaier, A. (1990). *Interval methods for systems of equations.* Cambridge University Press.

Oberguggenberger, M. and Fellin, W. (2008). "Reliability bounds through random sets: non-parametric methods and geotechnical applications." *Comput. Struct.*, 86(10), 1093–1101.

Penmetsa, R. C. and Grandhi, R. V. (2002). "Efficient estimation of structural reliability for problems with uncertain intervals." *Comput. Struct.*, 80, 1103–1112.

Pownuk, A. (2004). "Efficient method of solution of large scale engineering problems with interval parameters." *Proc. NSF workshop on reliable engineering computing*, R. L. Muhanna and R. L. Mullen, eds., Savannah, GA.

Qiu, Z. and Elishakoff, I. (1998). "Antioptimization of structures with large uncertain-but-non-rand parameters via interval analysis." *Compt. Meth. Appl. Mech. Engrg.*, 152, 361–372.

Qiu, Z., Yang, D., and Elishakoff, I. (2008). "Probabilistic interval reliability of structural systems." *Int. J. Solids & Structures*, 45, 2850–2860.

Rao, S. S. and Berke, L. (1997). "Analysis of uncertain structural systems using interval analysis." *AIAA J.*, 35(4), 727–735.

Rohn, J. and Rex, G. (1998). "Enclosing solutions of linear equations." *SIAM J. Numer. Anal.*, 35(2), 524–539.

Rump, S. M. (1990). "Rigorous sensitivity analysis for systems of linear and nonlinear equations." *Math. Comp.*, 54(190), 721–736.

Sun microsystems (2002). *Interval arithmetic in high performance technical computing.* Sun microsystems. (A White Paper).

Tonon, F., Bae, H.-R., Grandhi, R., and Pettit, C. (2006). "Using random set theory to calculate reliability bounds for a wing structure." *Struct. & Infrastruct. Engrg.*, 2, 191–200.

Williamson, R. (1989). "Probabilistic arithmetic," PhD thesis, University of Queensland.

Zhang, H. (2005). "Nondeterministic linear static finite element analysis: an interval approach," PhD thesis, Georgia Institute of Technology.

Ziemian, R. D. (1990). "Advanced methods of inelastic analysis in the limit states design of steel structures," PhD thesis, Cornell Univ.