Discrete Robust Design Optimization of Stochastic Dynamical Systems

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Abstract. In this paper attention is directed to the reliability-based optimization of uncertain structural systems under stochastic excitation involving discrete sizing type of design variables. The reliability-based optimization problem is formulated as the minimization of an objective function subject to multiple reliability constraints. The probability that design conditions are satisfied within a given time interval is used as a measure of system reliability. The objective function and the reliability constraints are approximated by using a hybrid form of linear and reciprocal approximations. The approximations are combined with an efficient sensitivity analysis to generate explicit expressions of the reliability constraints in terms of the design variables. The explicit approximate primal problems are solved by an appropriate discrete optimization scheme. A numerical example showing the efficiency and effectiveness of the approach reported herein is presented.

Keywords: approximation concepts, discrete optimization, reliability-based optimization, sensitivity analysis, uncertain systems

1. Introduction

For many structural optimization problems the design variables must be selected from a list of discrete values. For example, cross-sectional areas of truss members have to be chosen in general from a list of commercially available member sizes. In fact, design variables must be considered as discrete in a large number of practical design situations. In the reliability-based optimization literature that deals with stochastic dynamical systems little attention has been given to dealing with discrete design variables. The optimal design of structural systems under stochastic loading such as seismic excitations, water wave excitations, wind excitations, traffic loadings, etc., is usually carried out by considering continuous design variables. In this work attention is directed to discrete robust reliability-based optimization of structural systems under stochastic excitation. All uncertainties involved in the problem (system parameters and loading) are considered explicitly during the design process. Thus, final designs are robust in the sense that the optimization scheme accounts for the uncertainty in the system parameters as well as the uncertainty in the excitation. The reliability-based optimization problem is formulated as the minimization of an objective function subject to multiple reliability constraints. The probability that design conditions are satisfied within a given time interval is used as measure of system reliability. The basic mathematical programming statement of the structural optimization problem is converted into a sequence of explicit approximate primal problems of separable form. For this purpose, the objective function and the reliability constraints are approximated by using a hybrid form of linear and reciprocal approximations. Specif-

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ically, a conservative approximation is considered in the present formulation (Prasad, 1983; Haftka and Gürdal, 1992). The approximations are combined with an efficient simulation technique to generate explicit expressions of the reliability constraints in terms of the discrete design variables. The explicit approximate primal problems can be solved either by standard methods that treat the problem directly in the primal variable space (branch and bound techniques, combinatorial methods, evolution-based optimization techniques, etc.) (Tomlin, 1970; Kovács, 1980; Schrage, 1989; Goldberg, 1989) or by dual methods (Haftka and Gürdal, 1992; Jensen and Beer, 2009).

First, the primal optimization problem and the approximate primal problem are presented. Next, the corresponding solution of the approximate primal problem is considered. The proposed optimization process is presented and some implementation aspects are discussed. The application of the method to discrete robust reliability-based optimization of structural systems under stochastic loading is then considered. Finally, a numerical example is presented.

2. Problem Formulation

Consider a structural optimization problem defined as the identification of a vector $\{x\}$ of design variables to minimize an objective function , that is

$$\text{Minimize } F(\{x\}) \tag{1}$$

subject to the design constraints

$$h_j(\{x\}) \leq 0 \;\;,\;\; j = 1, ..., n_c \tag{2}$$

with side constraints

$$x_i \in X_i = \{\bar{x}_i^l, l = 1, ..., n_i\}, \; i \in I_D \tag{3}$$

where I_D denotes the set of indices for the discrete design variables and X_i represents the set of available discrete values for the design variable $x_i, i \in I_D$, listed in ascending order.

3. Approximation Concepts

The solution of the structural optimization problem given by Eqs. (1-3) is obtained by transforming it into a sequence of sub-optimization problems having a simple explicit algebraic structure. For this purpose, the objective and the constraint functions are approximated by using a hybrid form of linear and reciprocal approximations. In particular, a conservative approximation is considered in the present formulation (Prasad, 1983; Fleury and Braibant, 1986; Haftka and Gürdal, 1992; Jensen, 2006). Let $p(\{x\})$ be a generic performance function, i.e. the objective or constraint functions, and $\{x^0\}$ a point in the design space. The function $p(\{x\})$ is approximated about the point $\{x^0\}$ as

$$p(\{x\}) \approx \tilde{p}(\{x\}) \;=\; p(\{x^0\}) + \sum_{(+)} \frac{\partial p(\{x^0\})}{\partial x_i}(x_i - x_i^0) + \sum_{(-)} \frac{\partial p(\{x^0\})}{\partial x_i} \frac{x_i^0}{x_i}(x_i - x_i^0) \;, \tag{4}$$

where $\sum_{(+)}$ and $\sum_{(-)}$ mean summation over the variables belonging to group (+) and (-), respectively. Group (+) contains the variables for which $\partial p/\partial x_i(\{x^0\})$ is positive, and group (-) includes the remaining variables. The expansion given by Eq. (4) corresponds to a linear approximation in terms of the direct variables (x_i) for the variables belonging to group (+), and a linear approximation in terms of the reciprocal variables ($1/x_i$) for the variables belonging to group (-). An attractive property of this mixed linearization, called convex linearization, is that it yields the most conservative approximation among all possible combinations of direct/reciprocal variables (Haftka and Gürdal, 1992). In addition, the conservative approximation is a convex separable function.

4. Approximate Primal Problem

Applying the above linearization approach to the quantities involved in the optimization problem (1-3) yields the following approximate primal problem

Minimize

$$\sum_{(i^+)} \frac{\partial F(\{x^0\})}{\partial x_i} x_i - \sum_{(i^-)} \frac{\partial F(\{x^0\})}{\partial x_i} \frac{(x_i^0)^2}{x_i} \tag{5}$$

subject to

$$\sum_{(i_j^+)} \frac{\partial h_j(\{x^0\})}{\partial x_i} x_i - \sum_{(i_j^-)} \frac{\partial h_j(\{x^0\})}{\partial x_i} \frac{(x_i^0)^2}{x_i} \leq \bar{h}_j \ , \quad j = 1, ..., n_c \tag{6}$$

where

$$\bar{h}_j = \sum_{(i_j^+)} \frac{\partial h_j(\{x^0\})}{\partial x_i} x_i^0 - \sum_{(i_j^-)} \frac{\partial h_j(\{x^0\})}{\partial x_i} x_i^0 - h_j(\{x^0\}), \tag{7}$$

with side constraints

$$x_i \in X_i = \{\bar{x}_i^l, l = 1, ..., n_i\}, \ i \in I_D \tag{8}$$

where the groups (i^+) and (i_j^+) contain the variables for which the partial derivatives of the objective function and constraint functions are positives, respectively, and the groups (i^-) and (i_j^-) include the remaining variables. As before I_D denotes the set of indices for discrete design variables, and X_i represents the set of available discrete values for the design variable $x_i, i \in I_D$, listed in ascending order. It is noted that for the purpose of constructing the approximations of the objective and constraint functions all variables are assumed to be continuous. The approximate primal problem can be solved by any optimization algorithm that treats discrete design variables. They can be solved either by standard methods that treat the problem directly in the primal variable space or by dual methods. A genetic-based optimization algorithm is selected in this work (Goldberg, 1989).

5. Optimal Design Process

5.1. Algorithm Description

The optimization scheme is implemented as follows:

1. Start with an initial feasible design $\{x^k\}, k = 0$. Such design can be obtained in general from physical considerations or engineering criteria. If this is not the case several techniques are available for finding an initial feasible design (Jensen et al, 2009(b)).

2. The objective function $F(\{x\})$ and the constraint functions $h_j, j = 1, ..., n_c$ are approximated by using the convex linearization approach about the current design point $\{x^k\}$. At this step, all design variables are assumed to be continuous.

3. Formulate the explicit approximate primal optimization problem (5-8).

4. Solve the explicit approximate primal optimization problem by an appropriate discrete optimization scheme. A genetic-based optimization algorithm is used in the present implementation. The new point in the design space $\{x^*\}$ is used as the current design for the next cycle, that is, $\{x^{k+1}\} = \{x^*\}$. Set $k = k + 1$, and go to step 2.

5. The process is continued until some convergence criterion is satisfied.

5.2. Implementation Aspects

The approximations of the objective and constraint functions tend to be locally conservative because they result from the mixed linearization introduced in Section 3. This feature of the approximations implies that starting from an initial feasible design, the feasible domains corresponding to the explicit approximate primal optimization problems are generally inside the feasible domain of the original problem. In this manner, the process has the tendency to generate a sequence of steadily improved feasible designs. It is noted that the conservative approximation scheme is not guaranteed to be conservative in an absolute sense. That is, it is not known that the approximations are more conservative than the exact functions, which are unknown. It is also noted that the convergence of the above sequential approximate optimization technique to the solution of the original problem is not guaranteed from a strict mathematical point of view. However, this methodology has been found useful in a large number of structural optimization applications, including the cases considered in this study. In fact, for all the examples carried out by the authors the proposed sequential approximate optimization scheme converges within few iterations. This level of effectiveness is obtained when the curvatures of the iso-constraint curves in the design space are not too large and when two or more constraints are active at the final design. For more general cases methods based on strict/relaxed conservatism, local response surfaces, trust regions or line search are more appropriate (Moré, 1982; Bucher and Bourgund, 1990; Alexandrov et al, 1998; Svanberg, 2002; Agarwal and Renaud, 2004; Jensen and Catalan, 2007; Jensen et al, 2009(a))

6. Application to Reliability-Based Optimization

6.1. Formulation

In the context of reliability-based optimization of structural systems under stochastic excitation the design constraints can be written as

$$h_j(\{x\}) = P_{F_j}(\{x\}) - P^*_{F_j} \leq 0 \;\;,\;\; j = 1, ..., n_c \tag{9}$$

where $P_{F_j}(\{x\})$ is the probability of the failure event F_j evaluated at the design $\{x\}$, and $P^*_{F_j}$ is the corresponding target failure probability. The failure probability function $P_{F_j}(\{x\})$ evaluated at the design $\{x\}$ can be expressed in terms of the multidimensional probability integral

$$P_{F_j}(\{x\}) = \int_{\Omega_{F_j}} q(\{\theta\})p(\{z\})d\{z\}d\{\theta\} \;\;,\; j = 1, ..., n_c \tag{10}$$

where Ω_{F_j} is the failure domain corresponding to the failure event F_j and defined in terms of the performance function κ_j as

$$\Omega_{F_j} = \{\{\theta\}, \{z\} \mid \kappa_j(\{x\}, \{\theta\}, \{z\}) \leq 0\}. \tag{11}$$

The vectors $\{\theta\}, \theta_i, i = 1, ..., n_U$, and $\{z\}, z_i, i = 1, ..., n_T$ represent the vector of uncertain structural parameters and the random variables that specify the stochastic excitation, respectively. The uncertain structural parameters $\{\theta\}$ are modeled using a prescribed probability density function $q(\{\theta\})$ while the random variables $\{z\}$ are characterized by a probability function $p(\{z\})$. These functions indicate the relative plausibility of the possible values of the uncertain parameters $\{\theta\} \in \Omega_{\{\theta\}} \subset R^{n_U}$ and random variables $\{z\} \in \Omega_{\{z\}} \subset R^{n_T}$, respectively. The failure probability functions $P_{F_j}(\{x\}), j = 1, ..., n$ account for the uncertainty in the system parameters as well as the uncertainties in the excitation. For systems under stochastic excitation the probability that design conditions are satisfied within a particular reference period (first excursion probability) provides a useful reliability measure. The failure events $F_j, j = 1, ..., n_c$ are defined as

$$F_j(\{x\}, \{\theta\}, \{z\}) = \max_{i=1,...,n_j} \max_{t \in [0,T]} \mid s^i_j(t, \{x\}, \{\theta\}, \{z\}) \mid \geq s^{i^*}_j \tag{12}$$

where $[0, T]$ is the time interval, $s^i_j(t, \{x\}, \{\theta\}, \{z\}), j = 1, ..., n_c, i = 1, ..., n_j$ are the response functions associated with the failure criterion F_j, and $s^{i^*}_j$ is the corresponding critical threshold level. The response functions $s^i_j(t, \{x\}, \{\theta\}, \{z\}), j = 1, ..., n_c, i = 1, ..., n_j$ are obtained from the solution of the equation of motion that characterizes the structural model.

6.2. Approximate Reliability Constraints

The failure probability functions $P_{F_j}(\{x\}), j = 1, ..., n_c$ are represented using approximate functions dependent on the design variables. In particular, the construction of the approximate optimization problems is based on the approximation of transformed failure probability functions. Specifically, the following transformation is considered

$$h_{F_j}(\{x\}) = \ln[P_{F_j}(\{x\})], \quad j = 1, ..., n_c \tag{13}$$

The transformed failure probability functions are approximated globally by using the convex linearization previously defined. That is,

$$\tilde{h}_{F_j}(\{x\}) \;=\; h_{F_j}(\{x^0\}) + \sum_{(i_j^+)} \frac{\partial h_{F_j}(\{x^0\})}{\partial x_i}(x_i - x_i^0) + \sum_{(i_j^-)} \frac{\partial h_{F_j}(\{x^0\})}{\partial x_i}\frac{x_i^0}{x_i}(x_i - x_i^0), \quad j = 1, ..., n_c \tag{14}$$

where $\{x^0\}$ is a point in the design space, and $\sum_{(i_j^+)}$ and $\sum_{(i_j^-)}$ mean summation over the variables belonging to group (i_j^+) and (i_j^-), respectively. As before, group (i_j^+) contains the variables for which $\partial h_{F_j}(\{x^0\})/\partial x_i$ is positive, and group (i_j^-) includes the remaining variables. Numerical experience has shown that the above approximation scheme for the transformed failure probability functions is adequate in the context of this study.

7. Reliability and Sensitivity Estimation

7.1. Reliability Assessment

The reliability constraint functions $h_j(\{x\}), j = 1, ..., n_c$ defined in (13) are given in terms of the first excursion probability functions $P_{F_j}(\{x\}), j = 1, ..., n_c$. Subset simulation (Au and Beck, 2001) is adopted in this formulation for the purpose of estimating the corresponding failure probabilities during the design process. In the approach, the failure probabilities are expressed as a product of conditional probabilities of some chosen intermediate failure events, the evaluation of which only requires simulation of more frequent events. Therefore, a rare event simulation problem is converted into a sequence of more frequent event simulation problems. For example, the failure probability $P_{F_j}(\{x\})$ can be expressed as the product

$$P_{F_j}(\{x\}) \;=\; P(F_{j,1}(\{x\})) \prod_{k=1}^{m_{F_j}-1} P(F_{j,k+1}(\{x\})/F_{j,k}(\{x\})) \,, \tag{15}$$

where $F_{j,m_{F_j}}(\{x\}) = F_j(\{x\})$ is the target failure event and $F_{j,m_{F_j}}(\{x\}) \subset F_{j,m_{F_j}-1}(\{x\}) \; ... \subset F_{j,1}(\{x\})$ is a nested sequence of failure events. Equation (15) expresses the failure probability $P_{F_j}(\{x\})$ as a product of $P(F_{j,1}(\{x\}))$ and the conditional probabilities $P(F_{j,k+1}(\{x\})/F_{j,k}(\{x\})), k = 1, ..., m_{F_j} - 1$. It is seen that, even if $P_{F_j}(\{x\})$ is small, by choosing m_{F_j} and $F_{j,k}(\{x\}), k = 1, ..., m_{F_j} - 1$ appropriately, the conditional probabilities can still be made sufficiently large, and therefore they can be evaluated efficiently by simulation because the failure events are more frequent. The intermediate failure events are chosen adaptively using information from simulated samples so that they correspond to some specified values of conditional failure probabilities (Au and Beck, 2001).

7.2. Gradient Estimation

It is clear that the convex approximation of the reliability constraints requires the estimation of the gradient of the transformed failure probability functions. In what follows, a methodology for estimating the sensitivity of the transformed failure probability functions in terms of the design variables is presented.

7.2.1. *Performance Function Approximation*

Recall that the failure domain Ω_{F_j} for a given design $\{x\}$ is defined as

$$\Omega_{F_j} = \{\{\theta\}, \{z\} \mid \kappa_j(\{x\}, \{\theta\}, \{z\}) \leq 0\}. \tag{16}$$

If $\{x_k\}$ is the current design, the performance function κ_j is approximated in the vicinity of the current design as

$$\bar{\kappa}_j(\{x\}, \{\theta\}, \{z\}) = \kappa_j(\{x_k\}, \{\theta\}, \{z\}) + \{\delta_j\}^T \{\Delta x\} \tag{17}$$

where $\{x\} = \{x_k\} + \{\Delta x\}$. For samples $(\{\theta_i\}, \{z_i\}), i = 1, ..., M$ near the limit state surface, that is, $\kappa_j(\{x\}, \{\theta_i\}, \{z_i\}) \approx 0$ the performance function is evaluated at points in the neighborhood of $\{x_k\}$. These points are generated as

$$\{x_{lk}\} - \{x_k\} = \{\Delta x\} = \frac{\{\xi_l\}}{\parallel \{\xi_l\} \parallel} R \ , \ l = 1, ..., N = Q \times M \tag{18}$$

where the components of the vector $\{\xi_l\}$ are independent, identically distributed standard Gaussian random variables, N and Q positive integers and R is a user-defined small positive number. This number defines the radius of the hypersphere $\{\xi_l\}/ \parallel \{\xi_l\} \parallel R$ centered at the current design $\{x_k\}$. The coefficients $\{\delta_j\}$ of the approximation (17) are computed by least squares. To this end, the following set of equations is generated

$$\begin{aligned} \kappa_j(\{x_{lk}\}, \{\theta_i\}, \{z_i\}) = \kappa_j(\{x_k\}, \{\theta_i\}, \{z_i\}) + \{\delta_j\}^T \frac{\{\xi_l\}}{\parallel \{\xi_l\} \parallel} R \\ l = i + (q-1) \times M \ , q = 1, ..., Q \ , i = 1, ..., M \end{aligned} \tag{19}$$

Since the samples $(\{\theta_i\}, \{z_i\}), i = 1, ..., M$ are chosen near the limit state surface the approximate performance function $\bar{\kappa}_j$ is representative of the behavior of the failure domain Ω_{F_j} in the vicinity of the current design $\{x_k\}$.

7.2.2. *Sensitivity of Transformed Failure Probability Function*

The gradient of the j-th transformed failure probability function at $\{x_k\}$ is given by

$$\left. \frac{\partial h_{F_j}(\{x\})}{\partial x_l} \right|_{\{x\}=\{x_k\}} = \frac{1}{P_{F_j}(\{x_k\})} \times \left. \frac{\partial P_{F_j}(\{x\})}{\partial x_l} \right|_{\{x\}=\{x_k\}} \tag{20}$$

On the other hand, the gradient of the j-th failure probability function can be estimated by means of the limit:

$$\left.\frac{\partial P_{F_j}(\{x\})}{\partial x_l}\right|_{\{x\}=\{x_k\}} = \lim_{\Delta x_l \to 0} \frac{P_{F_j}(\{x_k\}+\{u(l)\}\Delta x_l) - P_{F_j}(\{x_k\})}{\Delta x_l}, \quad l=1,\ldots,n_d \tag{21}$$

where $\{u(l)\}$ is a vector of length n_d with all entries equal to zero, except by the l-th entry, which is equal to one. Considering the definition of failure probability in Eq. (10) and defining the normalized demand D_j as

$$D_j(\{x_k\},\{\theta\},\{z\}) = 1 - \kappa_j(\{x_k\},\{\theta\},\{z\}) \tag{22}$$

it can be shown that Eq.(21) can be rewritten as (Valdebenito and Schuëller, 2009)

$$\left.\frac{\partial P_{F_j}(\{x\})}{\partial x_l}\right|_{\{x\}=\{x_k\}} \approx \lim_{\Delta x_l \to 0} \left(\frac{P[D_j(\{x_k\},\{\theta\},\{z\}) \geq 1 + \delta_{lj}\Delta x_l]}{\Delta x_l} - \frac{P[D_j(\{x_k\},\{\theta\},\{z\}) \geq 1]}{\Delta x_l} \right), \quad l=1,\ldots,n_d \tag{23}$$

where $P[\cdot]$ denotes probability of occurrence. Thus, the calculation of the derivatives reduces to estimating the probability that the normalized demand (for a fixed design) exceeds two different threshold levels. These probabilities can be readily obtained from a single reliability analysis performed using, e.g. Subset Simulation. If the failure probability is approximated as an explicit function of the normalized demand as

$$\begin{aligned} P\left[D_j(\{x_k\},\{\theta\},\{z\}) \geq D_j^*\right] &= P_{F_j}(D_j^*) \\ P_{F_j}(D_j^*) &\approx e^{\psi_0 + \psi_1 (D_j^* - 1)}, \\ D_j^* &\in [1-\epsilon, 1+\epsilon] \end{aligned} \tag{24}$$

where D_j^* is a threshold of the normalized demand and ϵ represents a small tolerance, then Eq. (23) can be simplified to

$$\left.\frac{\partial P_{F_j}(\{x\})}{\partial x_l}\right|_{\{x\}=\{x_k\}} \approx \lim_{\Delta x_l \to 0} \frac{e^{\psi_0 + \psi_1 \delta_{lj}\Delta x_l} - e^{\psi_0}}{\Delta x_l} = \psi_1 \delta_{lj} P_{F_j}(\{x_k\}) \quad l=1,\ldots,n_d \tag{25}$$

and therefore

$$\left.\frac{\partial h_{F_j}(\{x\})}{\partial x_l}\right|_{\{x\}=\{x_k\}} \approx \psi_1 \delta_{lj} \quad l=1,\ldots,n_d \tag{26}$$

where δ_{jl} is the l-th element of the vector $\{\delta_j\}$, and all other terms have been previously defined. Numerical experience has shown that the above approximation is adequate in the context with the proposed optimization scheme. Issues such as the number of points required for performing least square (Q and M), and the generation of design points in the vicinity of the current design (calibration of the radius R) are discussed in (Valdebenito and Schuëller, 2009).

8. Numerical Example

8.1. Description

The finite element structural model shown in Figures (1) and (2) is considered in the numerical example. All floors have a constant height equal to 3.2 m, leading to a total height of 12.8 m. For a given floor all columns are assumed to be equal and their specifications are given in Table I. Additionally, twenty four tubular steel brace elements are placed in axis D, eight in axis 3, and eight in axis 6 (see Figure (2)), with nominal material properties $E = 2.1 \times 10^{11}$ N/m^2, and weight density $\rho = 7.42$ ton/m^3. It is assumed that each floor may be represented sufficiently accurate as rigid within the $x-y$ plane when compared with the flexibility of the columns. Hence, each floor can be represented by three degrees of freedom, i.e. two translatory displacements in the direction of the x and y axis, and a rotational displacement about the z axis. The associated masses $m_x = m_y$ and m_z are taken as constant for all floors 3.15×10^5 kg and 5.68×10^7 kg m^2, respectively. The Young's modulus E and the modal damping ratios of the structural model ζ_i are treated as uncertain system parameters. The Young's modulus is modelled by a Gaussian random variable with most probable value $\bar{E} = 2.1 \times 10^{11}$ N/m^2, and coefficient of variation of 10%, while the damping ratios are modelled by independent Log-normal random variables with mean values $\bar{\zeta}_i = 0.03$ and coefficients of variation of 10%.

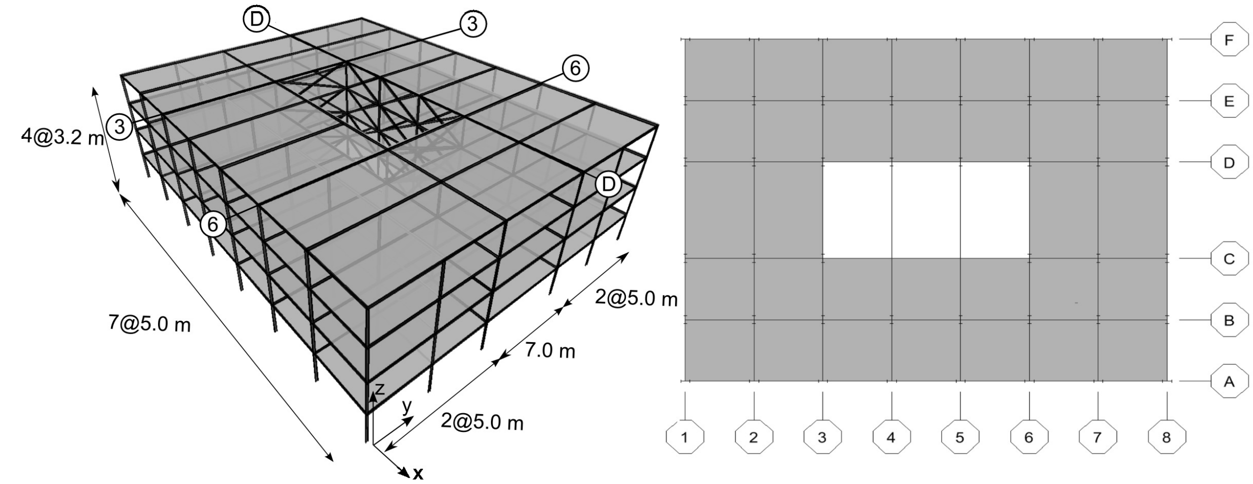

Figure 1. Isometric and plan view of the finite element model

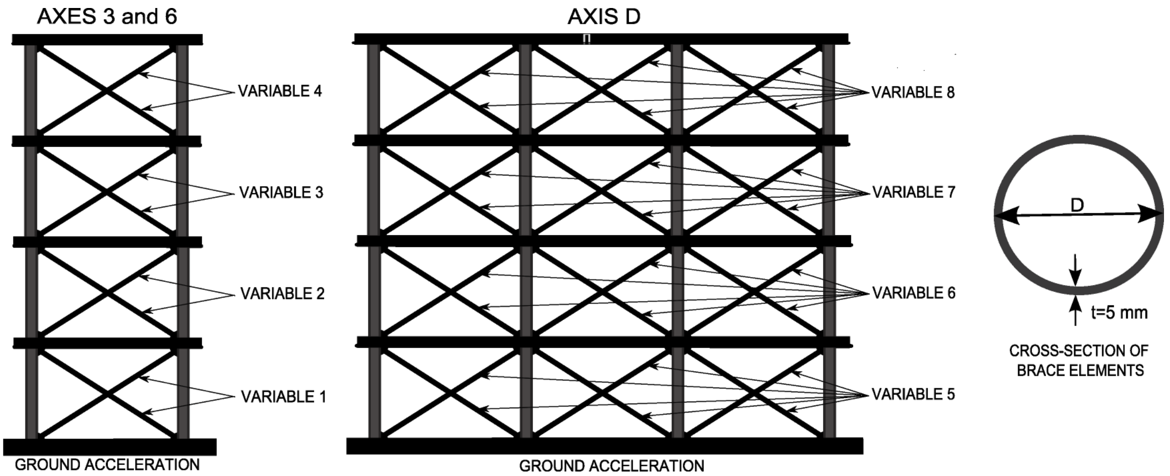

Figure 2. Axes 3, 6 and D of the structural model.

8.2. Stochastic Excitation

The structural model is excited horizontally by a bi-directional ground acceleration applied at 45 with respect to the x axis. The induced ground acceleration is modelled as a non-stationary filtered white noise process. The filter is characterized by the first-order differential equation

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\Omega_1^2 & -2\xi_1\Omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_1^2 & 2\xi_1\Omega_1 & -\Omega_2^2 & -2\xi_2\Omega_2 \end{pmatrix} \times \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} + \begin{pmatrix} 0 \\ w(t)e(t) \\ 0 \\ 0 \end{pmatrix} , \tag{27}$$

and the ground acceleration is defined as

$$a(t) = \Omega_1^2 y_1(t) + 2\xi_1\Omega_1 y_2(t) - \Omega_2^2 y_3(t) - 2\xi_2\Omega_2 y_4(t) , \tag{28}$$

where $w(t)$ denotes white noise and $e(t)$ denotes an envelope function. The values $\Omega_1 = 15.6$ rad/s, $\xi_1 = 0.8$, $\Omega_2 = 0.6$ rad/s, and $\xi_2 = 0.995$, and white noise intensity $I = 0.31$ m^2/s^3 have been used. The frequencies Ω_1 and Ω_2 correspond to the dominant and lower-cutoff frequency of the spectrum, respectively, and ξ_1 and ξ_2 are the damping parameters associated with the dominant and lower-cutoff frequency, respectively. The envelope function is defined as

$$e(t) \;=\; \frac{e^{-0.2t} - e^{-0.4t}}{\max(e^{-0.2t} - e^{-0.4t})} \quad , \;\; 0 < t < 20 \; s, \tag{29}$$

The sampling interval and the duration of the excitation are taken as $\Delta t = 0.01$ s and $T = 20$ s, respectively. Then, the discrete-time white noise sequence $\omega(t_j) = \sqrt{I/\Delta t} z_j$ where $z_j, j = 1, ..., 2001$, are independent, identically distributed standard Gaussian random variables is considered in this case.

Table I. Specification of column elements

Floor	Type of Section
1	$W24 \times 207$
2	$W24 \times 207$
3	$W24 \times 162$
4	$W24 \times 131$

Table II. Available values for the design variables

D (in)	A (mm^2)	D (in)	A (mm^2)	D (in)	A (mm^2)	D (in)	A (mm^2)
4	1517	5	1916	6	2315	7	2714
4 1/8	1567	5 1/8	1966	6 1/8	2365	7 1/8	2764
4 1/4	1617	5 1/4	2016	6 1/4	2415	7 1/4	2814
4 3/8	1667	5 3/8	2066	6 3/8	2465	7 3/8	2864
4 1/2	1717	5 1/2	2116	6 1/2	2515	7 1/2	2914
4 5/8	1767	5 5/8	2166	6 5/8	2565	7 5/8	2964
4 3/4	1817	5 3/4	2216	6 3/4	2615	7 3/4	3014
4 7/8	1866	5 7/8	2265	6 7/8	2664	7 7/8	3063

8.3. Optimization Problem

The objective function for the optimization problem is the total weight of the brace elements (W). The design variables are the areas of the cross-sections of the steel brace elements. Each floor is linked to two design variables which are related to resistant planes in the x direction (axis D), and in the y direction (axes 3 and 6), respectively (see Figure (1)). Thus, the vector of design variables $\{x\}$ has eight components in this case. The available discrete values for the design variables (areas of tubular cross-sections) are presented in Table II. The design criteria are defined in terms of global buckling conditions. Eight failure events are considered in the optimization problem and they are defined as

$$F_j(\{x\},\{\theta\},\{z\}) \;=\; \max_{t\in[0,10]} |N_j(t,\{x\},\{\theta\},\{z\})| > N_j^* \;, j=1,...,8 \tag{30}$$

where $\{\theta\}$ is the vector of uncertain structural parameters (Young's modulus and the modal damping ratios), $\{z\}$ is vector of random variables that characterizes the ground acceleration, $N_j(t,\{x\},\{\theta\},\{z\})$ is the normal compressive axial load in the brace elements associated with the design variable number j, and N_j^* is the corresponding axial load for which global buckling occurs. The discrete reliability-based optimization problem is formulated as

$$\text{Minimize } W(\{x\}) \tag{31}$$

subject to

$$P_{F_j}(\{x\}) = P[F_j(\{x\},\{\theta\},\{z\})] \leq P^*_{F_j} \quad , \quad j = 1,...,8 \ , \ \ x_i \in X \quad , \quad i = 1,...,8 \tag{32}$$

where X is the set of available areas of the cross-sections of the brace elements, and $P^*_{F_j}$ is the target failure probability which is taken equal to 10^{-3}.

8.4. Results

The problem is solved by using the sequential approximate optimization approach previously described. Subset simulation is used to estimate the failure probabilities and their sensitivities. The final designs of the deterministic and uncertain models are given in Table III. In the deterministic model the Young's modulus and the modal damping ratios of the structural model are equal to their most probable values. The corresponding iteration histories of the optimization process in terms of the objective function and the reliability constraints (deterministic and uncertain model) are shown in Figures (3), (4) and (5), respectively. Note that the probability of the failure events at the initial design of the uncertain model is substantially larger than the corresponding probabilities of the deterministic model. This is due to the effect of the uncertain system parameters. Starting from a feasible initial design, the process converges in less than four iterations. Recall that each design cycle requires only one reliability analysis for every reliability constraint. Therefore, the entire optimization process takes few excursion probability and sensitivity estimations. This computational cost is substantially different for the case of direct optimization. In that case the number of excursion probability and sensitivity estimations increases dramatically with respect to the proposed approach. In direct optimization the excursion probabilities and their sensitivities need to be estimated for every change of the design variables during the optimization process. At the final design some of the reliability constraints are not active due to the fact that the design variables can only take discrete values. From Figures (3), (4) and (5) it is also seen that the method generates a series of steadily improved feasible designs that move toward the solution. This feature of the scheme is important from a practical viewpoint since the design process may be stopped at any stage still leading to acceptable feasible designs better than the initial feasible estimate.

From Table III it is observed that the dimensions of the brace elements at the final design of the system with uncertain structural parameters are greater than the corresponding elements of the model with deterministic parameters. Therefore the total weight of the uncertain model increases with respect to the weight of the deterministic model. The dimensions of the brace elements of the resistant axes 3 and 6 are smaller than the corresponding elements of axis D. This is reasonable since the structural model, without the diagonal elements, is more rigid in the y direction. The importance of considering the effects of structural parameters uncertainty can be also illustrated from a constraint violation point of view. Table IV shows the probability of the failure events at the final designs of the deterministic model considering uncertainties. For example the probability of the first failure event obtained with the deterministic model is nine times the value of the target failure probability when the uncertainty in the system parameters is considered. Therefore, the

Table III. Final designs

Design variable	Initial design	Final design	
Area of cross-section		Uncertain model	Deterministic model
x_1 (mm^2)	2914	2565	2365
x_2 (mm^2)	2864	2365	2216
x_3 (mm^2)	2764	2365	2216
x_4 (mm^2)	2415	2016	1866
x_5 (mm^2)	3014	2814	2615
x_6 (mm^2)	2914	2615	2465
x_7 (mm^2)	2864	2615	2415
x_8 (mm^2)	2515	2216	2066
Structural weight (kg)	5491.3	4838.6	4505.8

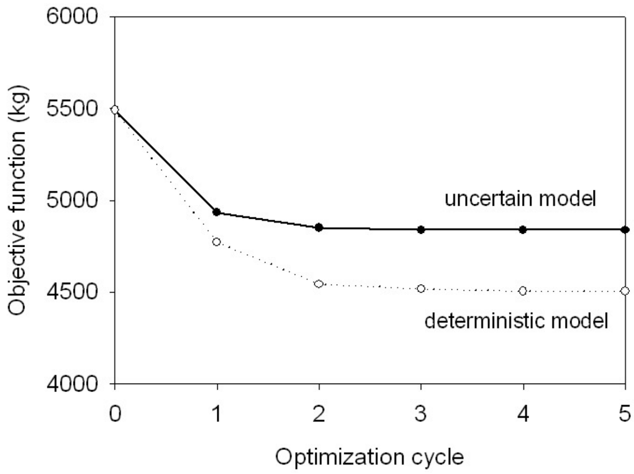

Figure 3. Iteration history in terms of the objective function. Deterministic and uncertain model.

optimal design of the deterministic model is not feasible. These results indicate the importance of considering the effect of system parameters uncertainty explicitly during the design process.

9. Conclusions

A methodology for robust reliability-based design optimization of stochastic non-linear systems involving discrete sizing type of design variables has been presented. The combined use of approximation concepts and genetic-based optimization schemes provides an efficient reliability-based design optimization capability for sizing problems which involve discrete design variables. Numerical

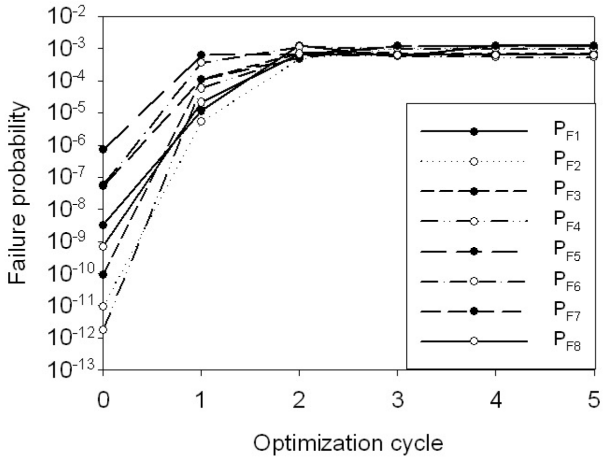

Figure 4. Iteration history in terms of the reliability constraints. Deterministic model.

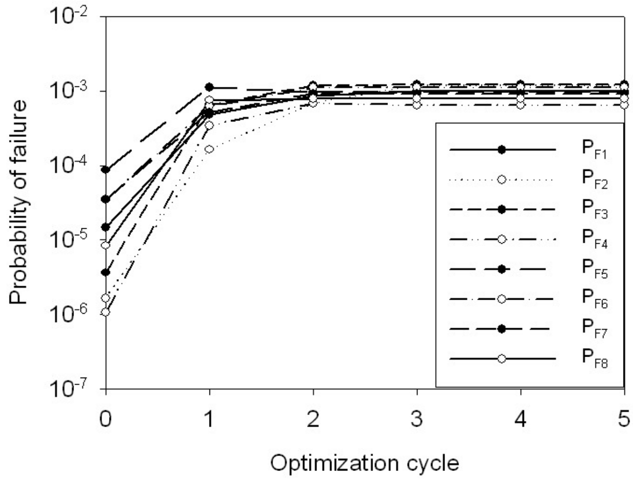

Figure 5. Iteration history in terms of the reliability constraints. Uncertain model.

results obtained from the example problem and from additional validation calculations have shown that the number of reliability estimations required during the optimization process is in general very small. This computational efficiency is essential especially for large complex finite element models since the computational efforts are reduced considerably in those cases. In addition, numerical results have shown that the approach generates a sequence of steadily improved feasible designs. This property is important from a practical view point since the optimization process can be stopped at any stage still leading to better designs than the initial feasible estimate. This is particularly attractive for dealing with involved problems such as reliability based optimization of dynamical systems under stochastic excitation. Numerical results have also shown that uncertainty in the

Table IV. Constraint violations

Failure event	Deterministic model	
	Probability of failure	Constraint violation ($P_{Fj}/P^*_{F_j}$)
F_1	9.0×10^{-3}	9.0
F_2	6.4×10^{-3}	6.4
F_3	6.7×10^{-3}	6.7
F_4	5.9×10^{-3}	5.9
F_5	7.2×10^{-3}	7.2
F_6	5.9×10^{-3}	5.9
F_7	7.6×10^{-3}	7.6
F_8	4.7×10^{-3}	4.7
target failure probability: $P^*_{F_j} = 10^{-3}$		

structural parameters may cause significant changes in the performance and reliability of systems subject to stochastic loading. For example, final designs obtained by deterministic models (in terms of structural characteristics) may become infeasible when the uncertainty in the system parameters is considered. In these situations the uncertainty in the specification of the structural properties should be properly accounted for during the design process. Based on the previous observations it is concluded that the proposed method represents a potential powerful tool in discrete variable structural optimization of stochastic dynamical systems.

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