Performance Measures for Robust Design and its applications

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Abstract. Taguchi methods are widely used for parameter design to make products more robust to variations of environmental conditions. Taguchi proposed various performance measures known as Signal-to-Noise (SN) ratios for evaluating the performance of signal-response systems. However, the SN ratios do not always minimize quadratic loss.

In this paper, we describe a generalized SN ratio based on average log quadratic loss and K loss. This SN ratio is an equivalent of a performance measure for evaluating a variation related to the dispersion parameter for data with positive values.

Keywords: loss function; robust design; signal-to-noise ratio; Taguchi method.

1. Introduction

The robust parameter design known as Taguchi's design of experiments aims to make the input/output relationships of systems and characteristic values respond stably to fluctuations in nosie factors (see, e.g., Taguchi et al (2004), Wu and Hamada (2000), Tsubaki and Kawamura (2008)). In other words, it makes system functions and product characteristics robust to noise factors that can be controlled in the laboratory but not in actual use situations or in an end-user environments.

To optimize this robustness, it is necessary to make performance measurements to evaluate the functional stability of the system. Taguchi's SN ratios were proposed for evaluating this sort of robustness. However, we do not use these SN ratios for two reasons. First, Taguchi's approach uses regression analysis (least squares) to minimize the data variation, regardless of how the variation of functions is evaluated. It is important to bear in mind that the basic concept of Taguchi methods is that noise is evaluated in terms of fluctuations in gfunctionalityh and not measurement errors. Therefore in this paper we directly evaluate the input/output ratio or the difference between the logarithmic input and output relative to the ideal proportional relationship. These methods are close to the original idea of Taguchi methods which is to directly evaluate variations in functionality or sensitivity. Second, our approach makes it possible to investigate robustness for data with collapsed orthogonality where it is not necessarily possible to acquire a strict design of experiments. In recent years the application of this technique to numerical design of experiments based on simulation has also been considered.

Taguchi's SN ratios are usually defined as the reciprocal of the mean square loss. In this paper, instead of a quadratic loss function, we define the SN ratio using other functions such as a

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quadratic log loss function and a K loss function. The SN ratios are a dimensionless performance measurement, and can thus be evaluated (compared) as an absolute value. Here, we postulate a multiplicative model. This is because in elementary applications of Taguchi methods, this model postulates standard conditions where the variation is zero when a characteristic has an average value of zero, especially for ratio scale data (data that has an absolute zero point). An SN ratio based on a K loss function or quadratic log loss function is a special case of a dispersion parameter or a generalized SN ratio according to Box (1988). We will also show that this dispersion parameter satisfies the PerMIA criteria proposed by Leon et al. (1987) and is closely connected with the two-step procedure.

2. Taguchi's SN ratio

2.1. Nominal-the-best SN ratio

This section introduces Taguchi's SN ratio in parameter design. A design parameter is a factor that influences the quality property, and whose value can be set by the user. In general, a system will be better if it has more are a lot of design parameters in the basic design (system selection). Noise factors are uncontrollable factors that cause variations in the quality property. The user can only select the main factor in the experiment because there are innumerably many noise factors. Noise factors can be tackled by performing observational studies using statistical analysis.

The basic idea behind parameter design is to find a design parameter that reduces the influence of the noise factor. To evaluate a design quantitatively, the user conducts an experiment that the system's u se and environmental conditions.

Taguchi's SN ratios are formulated as follows. Given an input M, we define the population mean $\mathrm{E}_N[Y|M]$ and population variance $\mathrm{var}_N[Y|M]$ in terms of the distribution of the noise factor N of output Y as follows:

$$\mathrm{E}_N\left[Y|M\right] = \mu(M|\theta) \tag{2.1}$$

$$\mathrm{var}_N\left[Y|M\right] = \phi(M|\theta)V(\mu(M|\theta)). \tag{2.2}$$

The reciprocal of this dispersion parameter ϕ is the generalized SN ratio proposed by (Box (1988)).

Noise factor N disturbs the ideal relation between characteristic y and signal factor M. In the Taguchi method, the level of the restrictor is fixed, N and M are allocated in a two-way layout, and y_{ij} is observed. Table I shows this data format.

Equations (2.1) and (2.2) can be calculated by using equations (2.3) and (2.4), respectively

$$\mathrm{E}_N\left[Y|M_i\right] \doteq \bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j \tag{2.3}$$

$$\mathrm{var}_N\left[Y|M_i\right] \doteq s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(y_j - \bar{y})^2 \tag{2.4}$$

Table I. Two-way layout of noise factor and signal factor

	M_1	$\cdots$	M_i	$\cdots$	M_m
N_1	y_{11}	$\cdots$	y_{1i}	$\cdots$	y_{1m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N_j	y_{j1}	$\cdots$	y_{ji}	$\cdots$	y_{jm}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N_n	y_{n1}	$\cdots$	y_{ni}	$\cdots$	y_{nm}

In Taguchi methods, the variance function $V(\mu)$ of equation (2.2) is assumed to be μ^2. The reciprocal of the dispersion parameter ϕ is called the nominal-the-best SN ratio, and is defined as follows:

$$\eta_T = \frac{1}{\phi(M_i|\theta)} = \frac{\mu^2}{\sigma^2} \tag{2.5}$$

Here, σ^2 is a population variance. This SN ratio is the function of the coefficient of variance.

Characteristic Y is assumed to be a continuous positive random variable from a population (μ, σ^2) with a population mean $\mathrm{E}[Y] = \mu$ and a target μ_{opt}. For the quadratic loss function, the risk function is the average squared-error loss, i.e.,

$$\begin{aligned} R &= E\left[(Y - \mu_{\text{opt}})^2\right] \\ &= (\mu - \mu_{\text{opt}})^2 + E[(Y-\mu)^2] \end{aligned} \tag{2.6}$$

We can make this dimensionless by dividing (2.6) by μ^2:

$$R' = \left(\frac{\mu_{\text{opt}}}{\mu} - 1\right)^2 + E\left[\left(\frac{Y}{\mu} - 1\right)^2\right] \tag{2.7}$$

Thus, the first step of the two-step optimization for a nominal-the-best application is to maximize the SN ratio (minimize the coefficient of variance μ/σ) and thereby minimize the variability around the average. The second step is to adjust the mean to the target μ_{opt}.

Equation (2.5) is estimated as

$$\eta_T = \frac{1}{\hat{\phi}(M_i|\theta)} = \frac{\bar{y}^2}{s^2} \tag{2.8}$$

The SN ratio in decibel (dB) is calculated as

$$\eta_T \text{ [dB]} = 10 \log_{10} \frac{\bar{y}^2}{s^2} \tag{2.9}$$

The performance measure is called the conditional SN ratio of the output for a given input M. For experimental data of the sort shown in Table I, a signal factor M_i can also cause the standard to fluctuate, so instead of the given M, it is useful to evaluate the robustness of the input/output functions as a whole. The SN ratios of so-called dynamic characteristics can be evaluated as discussed in the following section.

2.2. Dynamic SN ratio

In general, model adaptation in the statistical approach is performed by adapting a statistical model that approximates the input/output function. In other words, this approach involves using the data to determine the functional form that best fits a model that gives the functional form of the system f:

$$Y = f(M \mid \theta, N) + \varepsilon$$

On the other hand, this type of model adaptation approach is not used for the parameter design of dynamic characteristics. Specifically, by postulating a theoretical function $g(M)$ as the form that should be taken by the output transform $h(Y)$, it only evaluates the divergence from the viewpoint of a loss function. Here, the choice of theoretical function is a matter of system selection.

The real output characteristics are analyzed as follows:

$$h(Y) = B + g(M) + NF, \tag{2.10}$$

where NF is noise fluctuation component and B is the sensitivity. If the target of B is β_T, then the ideal characteristics $h_{\text{opt}}(Y)$ are given by

$$h_{\text{opt}}(Y) = \beta_T + g(M) \tag{2.11}$$

The divergence of the real output characteristics from the ideal characteristics can be formulated as the average loss as follows:

$$\mathrm{E}[(h(Y) - h_{\text{opt}}(Y))^2] = (B - \beta_T)^2 + V, \tag{2.12}$$

where V is the variance component of noise fluctuations originating from the fluctuation of noise factors.

The loss function of dynamic characteristics into which signal factors have been introduced includes the loss function of nominal-the-best characteristics as a special case if we assume the ideal functions $g(M) = 0$ and $h(Y) = Y$. Accordingly, evaluating the robustness of the dynamic characteristics can be described as a methodology for evaluating how much the sensitivity fluctuates with fluctuations in the noise factors.

The issue of stability in the input/output characteristics in the case where a signal factor M exists can be formulated as follows:

$$\mathrm{E}_N[B] = \beta(\theta) \tag{2.13}$$

$$\mathrm{var}_N[B] = \phi(\theta)V[\beta(\theta)] \tag{2.14}$$

If we take the above formulation as a premise, then the following two principles can be considered for the analysis of SN ratios of dynamic characteristics.

(i) Evaluation of estimation sensitivity robustness: For each level of the noise factor, the function $h(Y) = \beta + g(M)$ is applied taking Equations (2.7) and (2.8) into consideration, and the robustness of B is evaluated by statistically estimating the mean and variance of B.

(ii) Direct method: Taking the data $h(Y(M)) - g(M)$ as a characteristic value, the SN ratio is analyzed in the same way as a nominal-the-best characteristic. In this direct method, the level fluctuations of signal factors are themselves regarded as systematic noise factors.

The SN ratio analysis of dynamic characteristics that has been used in Taguchi methods differs from both (i) and (ii) above. Instead, it is a methodology that employs analysis of variance. Section 3 of this paper presents a detailed introduction of SN ratios that are closer to Taguchi's basic principles.

3. Performance Measures for Robust Parameter Design

3.1. Logarithmic SN ratio

In Taguchi methods, the target values of characteristic values are called nominal-the-best characteteristics when they are positive finite numbers, and the SN ratio is defined as the square of the reciprocal of the coefficient of variation. On the other hand, the SN ratio of nominal-the-best characteristics can be regarded as the reciprocal of the average squared-error loss, or to put it more accurately, the reciprocal of the average squared-error loss expressed as a dimensionless number.

Let's assume Y to be a continuous positive random variable, defined by the quadratic logarithm loss function:

$$L = \left(\log\left(\frac{Y}{\mu}\right)\right)^2 \tag{3.1}$$

The average loss is given by

$$c^2 \equiv R_L = E\left[\left(\log\left(\frac{Y}{\mu}\right)\right)^2\right], \tag{3.2}$$

where the parameter μ is the geometric mean of Y and satisfies $E[\log(Y/\mu)] = 0$. For a static parameter design problem with continuous output Y the SN ratio is

$$\eta_L = \frac{1}{c^2} \tag{3.3}$$

Using the data format in Table 2.1, the estimation of (3.3) is calculated by

$$\eta_T = \frac{1}{C^2}, \tag{3.4}$$

where

$$C^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}\left(\log\left(\frac{y_i}{\bar{y}_G}\right)\right)^2 \tag{3.5}$$

and $\bar{y}_G = (y_1 \cdot y_2 \cdots y_n)^{\frac{1}{n}}$ is the sample geometric mean. Also, since $E\left[C^2\right] = c^2, n \geq 2$, the unbiased estimator of c^2 is given by C^2. In this paper, the reciprocal of this estimator is defined as the SN ratio. The sample SN ratio (3.4) of nominal-the-best characteristics in Taguchi methods is sometimes defined by the respective unbiased estimators of the numerator and denominator of the population SN ratio. However, this statistic is not an unbiased estimator of the population SN ratio itself. On the other hand, our proposed approach it is an unbiased estimator of the population SN ratio. When the level of noise factors swings widely, it is possible for the conventional sample SN ratio to take a negative value, but the statistic C^2 is always guaranteed to be positive.

Parameter design based on dynamic characteristics is a methodology for declaring systems and functionalities as input/output functions, and determining the conditions under which these relationships approach the ideal functions. Here, the input is taken as the signal factor M, and the real output characteristic is represented as y. The relationship between the two is assumed to be a zero-point proportional model $y = \beta M$. Thus the average square logarithmic loss of $Y/(\beta M)$ for an arbitrary value of M is given by

$$E\left[\left(\log\left(\frac{Y}{\beta M}\right)\right)^2\right] \equiv c^2 \tag{3.6}$$

In section 2 it was assumed that $g(M) = \log M$, $h(y) = \log y$, and $\beta_T + \log M$ was postulated as the ideal function of output $\log Y$. This relationship is a proportional relationship $Y \propto M$ with a proportionality constant of $\exp(\beta_T)$.

The SN ratio of the dynamic characteristics for a zero-point proportional equation is defined as:

$$\gamma_L = \frac{1}{c^2} \tag{3.7}$$

and is called the population SN ratio of the dynamic characteristics based on the average logarithmic loss. It should be noted that the population SN ratio defined here is the same as for the nominal-the-best characteristics and is a dimensionless quantity. It therefore allows comparisons to be made between designs that use different physical mechanisms and different units for the signal factor M, and is meaningful as an absolute number. This contrasts with the population SN ratio of dynamic characteristics in conventional Taguchi methods where care must be taken when comparing systems with inherently different signal factors.

The noise factor N is what disturbs the ideal relationship of the characteristic y and signal M. In Taguchi methods, the level of the control factor is kept fixed while observing the characteristic factor y with N and M allocated in two-way layout as shown in Table 2.1.

For an observed value y_{ij} at $N_i M_j$, we assume that the following zero-point proportional model (multiplicative model):

$$y_{ij} = \beta M_j \cdot \varepsilon_{ij} \tag{3.8}$$

This is a model that normalizes the observed value y_{ij} by the signal factor M_j and expresses the main effects of the noise factor in the gradient β:

$$\frac{y_{ij}}{M_j} = \beta \cdot \varepsilon_{ij} \tag{3.9}$$

If we want to stress that this is an error of functionality, we can write it as follows:

$$y_{ij} = (\beta_i \cdot \varepsilon_{ij}) \cdot M_j \tag{3.10}$$

This is a model in which errors are attached to sensitivity (functionality), and which is used for evaluating the functionality variation β itself which is the basic idea of Taguchi methods. Compared with the usual additive model, it is clearly an error that causes variation in function and not a measurement error.

If we change the left side of Equation (3.9) from y_{ij}/M_j to y'_i again, then

$$y'_{ij} = \beta \cdot \varepsilon_{ij} \tag{3.11}$$

and it can be seen that this corresponds to the case of the nominal-the-best characteristic.

In this case the sample SN ratio is obtained by normalizing the data y_{ij} by the signal factor M_j to obtain the normalized data $y'_{ij}(= y_{ij}/M_j)$.

¿From here, the calculations are performed using Equation (3.4) in exactly the same way as for nominal-the-best characteristics. The corresponding sample SN ratio for dynamic characteristics is given by

$$\gamma_L = \frac{1}{C^2}, \tag{3.12}$$

where

$$C^2 = \frac{1}{nm-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\log \frac{y'_{ij}}{\bar{y'}_G} \right)^2 \tag{3.13}$$

Here, $\bar{y'}_G$ is the sample geometric mean of all the data y'_{ij} (see, Kawamura and Iwase (2009)).

3.2. K SN ratio

Let's assume characteristic Y to be a continuous positive random variable. The average K loss is given by

$$c^2 \equiv R_K = E\left[\left(\sqrt{\frac{Y}{\mu}} - \sqrt{\frac{\mu}{Y}} \right)^2 \right], \tag{3.14}$$

where parameter μ is the population arithmetic mean of Y. The reciprocal of (3.14) is

$$\eta_K = \frac{1}{c^2} \tag{3.15}$$

as was done in the nominal-the-best SN ratio. This SN ratio is a non-dimensional parameter. Using the data format in Table 2.1, (3.15) can be estimated as

$$\eta_T = \left(1 - \frac{1}{n}\right) \cdot \frac{1}{C^2} \tag{3.16}$$

Here, the statistic C^2 is given by

$$C^2 = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sqrt{\frac{y_j}{y_i}} - \sqrt{\frac{y_i}{y_j}}\right)^2$$
$$= \frac{\bar{y}}{\bar{y}_H} - 1,$$

where $\bar{y}_H$ is the sample harmonic mean. Thus, the following relation holds:

$$E\left[\left(\frac{n}{n-1}\right)C^2\right] = c^2, \quad n \geq 2$$

and the unbiased estimate of c^2 becomes $(n/(n-1))C^2$ (See, Kawamura et al. (2006), p.98 (A.5)).

Let's assume the input M and output Y relation to be a zero point proportional model $y = \beta M$. Then the average K loss of $Y/(\beta M)$ for an arbitrary M is given by

$$c^2 \equiv E\left[\left(\sqrt{\frac{Y}{\beta M}} - \sqrt{\frac{\beta M}{Y}}\right)^2\right] \tag{3.17}$$

Thus, the SN ratio of the dynamic characteristic is given by

$$\gamma_K = \frac{1}{c^2} \tag{3.18}$$

Next, let us assume that the observed values y_{ij} in N_iM_j conform to a zero point proportional model (3.8). If we assume that the observed values y_{ij} are normalized by the signal factors M_j, then this is the same as for the nominal-the-best characteristics. The SN ratio of the dynamic characteristic is given by

$$\gamma_K = \left(1 - \frac{1}{nm}\right)\frac{1}{C^2}, \tag{3.19}$$

where

$$C^2 \equiv \frac{\bar{y'}}{\bar{y'}_H} - 1 \tag{3.20}$$

4. Taguchi's approach to robust parameter design

4.1. Two-step optimization procedure

In parameter design, the SN ratio is usually optimized at the first stage to make the system stable. Then at the second stage the sensitivity is adjusted by means of control factors (adjustment factors) that do not affect the system's stability or robustness. In Taguchi methods, this approach is called a two-step procedure (Taguchi (1977)). In the previous section, we discussed Taguchi's SN ratios and SN ratios based on various different loss functions. These SN ratios are defined by dispersion parameters and measure the robustness performance of the system's input/output relationships and characteristic values. But there must also be a parameter, PerMIA that is independent of the sensitivity adjusted in the second stage of a two-step procedure (Leon, et al. (1987)). The idea of making the parameters independent or orthogonal was proposed by Cox and Reed (Cox and Reid (1987)). Determining the generalized SN ratio and variance function $V(\mu)$ are equivalent, which raises the question of how to postulate the variance function. Here, the SN ratio must exhibit the PerMIA condition of being independent of the average fluctuation of the system output.

We described a two-step optimization procedure using the generalized SN ratio. When the characteristics or sensitivity B by the noise factor is intentionally varied, it is assumed that the mean and variance are as follows:

$$\mathrm{E}_N[B] = \beta(\theta_1, \theta_2) \tag{4.1}$$

$$\mathrm{var}_N[B] = \phi(\theta_1)\mathrm{var}(\beta(\theta_1, \theta_2)) \tag{4.2}$$

Note that the control factor is divided into two groups θ_1 and θ_2, and that the dispersion parameter ϕ is only dependent on θ_1. In Taguchi methods, θ_1 is called the design parameter, and θ_2 is called the adjustment factor. The target value of the sensitivity is taken to be β_T, the loss of quality is minimized by the following two steps:

STEP1. $\phi(\theta_1^*) = \min_{\theta_1} \phi(\theta_1)$
STEP2. $\beta(\theta_1^*, \theta_2^*) = \beta_T$

In other words, the first stage searches for control factors that affect the SN ratio, selects their optimal levels, and makes the system robust by maximizing the SN ratio. The second stage then searches for control factors (i.e., adjustment factors) that affect the sensitivity without affecting the SN ratio, and sets their levels so that the sensitivity achieves the optimal value. If these optimal design parameters θ_1^*, θ_2^* exist, then the loss of quality is given by $\phi(\theta_1^*)\mathrm{var}(\beta_0)$. Also, if θ_1^*, θ_2^* exists, then it is not necessarily unique and this optimal solution may not always exist. To increase the possibility that this sort of solution exists, Taguchi methods adopt the idea of using complex nonlinear systems to increase the number of control factors.

This two-step procedure is the opposite of the normal method where robustness is achieved by matching the average value to the target. Specifically, it achieves robustness by adjusting the average value, and the level θ_1^* of the control factor that achieves robustness does not depend on the target value β_0. Although the target value may change if the user finds it necessary, this formulation has the distinct advantage that even when this sort of change in target occurs, it is always best to use the same value for the control factor. The target only depends on the second stage of the two-step procedure, which is the only part requiring adjustment. Or looked at from the opposite direction, robust design is independent of the target values and can be performed before the target values have been determined.

4.2. Minimization of average loss and two-step procedure

If the sensitivity B is a positive random variable given by a function of the design parameter θ and noise N, the average logarithmic loss due to the fluctuation of N can be written as follows:

$$R_L(\theta) = E_N\left[\left(\log\frac{B}{\beta_T}\right)^2\right] \tag{4.3}$$

Here β_T is a targeted value. By dividing the design parameter θ into two groups, $\theta = (\theta_1, \theta_2)$, $R_L(\theta_1, \theta_2)$ can be minimized through the following two-step procedure:

1. Find θ_1* that minimizes $P_L(\theta_1^*) = \min_{\theta_1} R_L(\theta_1, \theta_2)$.

2. Find θ_2^* that minimizes $R_L(\theta_1^*, \theta_1)$.

Now consider $f(\theta, N)$ to be a multiplicative model:

$$B = \beta(\theta_1, \theta_2)\varepsilon(N, \theta_1) \tag{4.4}$$

Here, we assume the moment conditions: $E\left[\log(\varepsilon(N,\theta_1))\right] = 0$ and $E\left[\log(\varepsilon(N,\theta_1))^2\right] = c^2(\theta_1)$. The mean (geometric mean) β is a function only of the design parameter $\theta = (\theta_1, \theta_2)$, and the noise ε is a function of N and θ_1, and is independent of θ_2. Here, $P(\theta_1) = c^2(\theta_1)$, and maximizing the SN ratio of the dynamic characteristics (Equation (3.7)) is equivalent to minimizing $P(\theta_1)$.

Let us discuss two-step optimality when average K loss is used (see, Kawamura and Iwase (2007)). When the sensitivity B is taken to be a positive-valued random variable given by a function of the design parameter $\boldsymbol{\theta}$ and noise $\boldsymbol{N}$, the average K loss due to fluctuation of $\boldsymbol{N}$ can be written as follows:

$$R_K(\theta) = E_N\left[\left(\sqrt{\frac{B}{\beta_T}} - \sqrt{\frac{\beta_T}{B}}\right)^2\right]$$

Let us consider $f(\theta, N)$ to be a multiplicative model (4.4) and $\varepsilon(N, \theta_1)$ has first-order and reciprocal moments $E\left[\varepsilon(N,\theta_1)\right] = 1$, $E\left[1/\varepsilon(N,\theta_1)\right] = c^2(\theta_1) + 1$.

Note that under model(4.4) and the average K loss,

$$R_K(\theta) = \frac{\beta(\theta_1,\theta_2)}{\beta_T} + \frac{\beta_T}{\beta(\theta_1,\theta_2)}\{c^2(\theta_1) + 1\} - 2 \tag{4.5}$$

Therefore, if we solve for $\partial R_K(\boldsymbol{d}, \boldsymbol{a})/\partial \boldsymbol{a} = 0$, we get

$$\beta(\theta_1, \theta_2^*(\theta_1)) = \beta_T\sqrt{c^2(\theta_1) + 1}, \tag{4.6}$$

where $\theta_2{}^*(\theta_1)$ is the value of $\boldsymbol{a}$ that yields $P(\boldsymbol{d}) = \min_{\boldsymbol{a}} R_K(\boldsymbol{d}, \boldsymbol{a})$. If we substitute (4.5) into (4.6), we obtain

$$P_K(\theta_1) = 2(\sqrt{c^2(\theta_1) + 1} - 1).$$

Hence, since $P_K(\theta_1)$ is a strictly decreasing function of the SN ratio (3.18), maximizing η_K is equivalent to minimizing $P_K(\theta_1)$. Thus, the abovementioned two-step optimization procedure for minimizing the average K loss is equivalent to the following procedure:

1. Find ${\theta_1}^*$ that maximizes the SN ratio (3.18).
2. Find ${\theta_2}^*$ such that $\beta({\theta_1}^*, {\theta_2}^*) = \beta_T\sqrt{c^2(\theta_1) + 1}$.

This proves the validity of the two-step procedure involving the SN ratio based on the average K loss.

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