A fuzzy finite element analysis technique for structural static analysis based on interval fields

Wim Verhaeghe, Maarten De Munck, Wim Desmet, Dirk Vandepitte and David Moens
Department of Mechanical Engineering, K.U.Leuven, 3000 Leuven, Belgium,
wim.verhaeghe@mech.kuleuven.be

Abstract. One of the main shortcomings of current fuzzy and interval finite element procedures is that mutual dependency between multiple uncertain model parameters cannot be included in the analysis. This limit is posed by the classical interval concept, where multi-dimensional interval quantities are generally treated as hypercubes, thus ignoring all possible dependency between vector components. For this reason, most literature on this subject focuses on one-dimensional output quantities.

In order to cope with this problem, this work discusses the application of the concept of interval fields for static analysis of uncertain mechanical structures in the context of fuzzy finite element analysis. The theoretic background of the concept is explained, and it is shown how it can be applied to represent dependency between parametric uncertainties.

Further, the paper concentrates on the calculation of interval fields resulting from static structural analysis. A procedure that enables the calculation of a joint representation of multiple output quantities of a single interval finite element problem while preserving the mutual dependency between the components of the output vector is introduced. This procedure is based on a projection of the original problem on the space composed by the classical static deformation patterns. This paper in particular introduces a novel projection in which the space of the classical deformation patterns is augmented with deviatoric parts. This novel projection leads to a better approximation of the results without a significant increase in computation time.

Finally, a numerical case study illustrates the procedure and validates the improved accuracy of the results obtained with the novel projection technique.

Keywords: fuzzy finite elements; fuzzy analysis; interval analysis; interval fields.

1. Introduction

In recent years a lot of research into the development of non-deterministic finite element methods has been done. This resulted in a variety of methods, ranging from the stochastic, to the non-probabilistic fuzzy and interval finite element method (see e.g. (D. Moens, 2005)). Two general difficulties arise in both families of methods (probabilistic and non-probabilistic): the problem of dependency and the problem of high calculation costs.

In the probabilistic methods the concept of random fields (E. Vanmarcke, 1993) is used to take into account the dependency between uncertain parameters. Using random fields requires the covariance function of the studied quantity to be known. It is doubted (D. Moens, 2006) that

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such information is available in a reliable way when the overall problem is dominated by non-determinism. In general using the probabilistic concepts requires a lot of knowledge on the variability or uncertainty before being applicable and usefull. The interval field representation presented here is a non-probabilistic concept for taking into account the dependencies.

Most of these methods (probabilistic and non-probabilistic) are computationally expensive due to the fact that at their core they have a deterministic problem that has to be solved multiple times to obtain the non-deterministic results. Generally speaking, this can be remedied in two ways: reducing the needed number of deterministic analysis or reducing the calculation cost of one deterministic analysis. The interval field representation can be categorised in the latter, while it also takes into account dependency between parametric uncertainties.

A general introduction to non-deterministic fields is given in the next section. The focus is on the non-probabilistic static finite element solution methods. The third section presents the interval field description of the FE output. In the fourth section a numerical case study illustrates the benefits of the novel projection technique (using as a basis the classical static deformation patterns, augmented with deviatoric parts) for interval field FE output.

2. Non-deterministic fields in the FEM

Firstly, the terminology of non-determinism and the concepts of dealing with it are briefly reviewed. Secondly, fields are introduced, with a focus on the static finite element method. In the last part of this section the influence of non-determinism on fields is presented as well as the general concept of dealing with non-deterministic fields using the interval field description.

2.1. Non-determinism

2.1.1. *Terminology*

Non-determinism has various causes. This paper applies the terminology proposed by Oberkampf (W. Oberkampf, 1999) and its refinements proposed by Moens (D. Moens, 2005).

Variability is defined as the variation which is inherent to the modelled physical system or the environment under consideration (e.g. manufacturing tolerances). *Uncertainty* is defined as a potential deficiency in any phase or activity of the modelling process that is due to a lack of knowledge (e.g. models for material damping). An *error* is defined as a recognisable deficiency in any phase of modelling or simulation that is not due to a lack of knowledge.

Moens proposes a refinement on the terms variability and uncertainty because they are not mutually exclusive. A variability is characterised by a range of possible values and the likelihood of each variable within this range. However, a variability can be an uncertainty too when no or limited information is available on this range of possible values or on the likelihood of each value. This is called an uncertain variability. A variability of which both the range and the likelihood of each value within this range is known is called a certain variability. The same distinction can be made for uncertainties. An uncertainty that by nature has a deterministic value but cannot be reliably modelled as such due to a lack of knowledge is called an invariable uncertainty. On the other hand, an uncertain parameter that exhibits variability is called a variable uncertainty, although in general the effect of the variability is negligible compared to the effect of the uncertainty.

2.1.2. *Probabilistic concept*

A probabilistic quantity x is mathematically characterised by its probability density function (PDF), which can be summarised by its mean value and variance. The probabilistic analysis is the perfect tool for analysing models with certain variabilities. An uncertain variability has to be modelled by every PDF consistent with the limited information available. An invariable uncertainty is modelled with a subjective PDF chosen by the analyst. In the latter two cases it is important to acknowledge the subjectivity in the results of an analysis.

Probabilistic results are not always needed; most of the time only an improvement in the behaviour of a physical model under non-deterministic influences is persued. This can be achieved by the probabilistic concept, but such probabilistic information is not always primordial.

The still improving computational power makes the issue of the probabilistic methods (e.g. Monte Carlo simulation) being computationally expensive less important.

2.1.3. *Non-probabilistic concepts*

An interval scalar $x^I = [x_{min}\ x_{max}] = [\underline{x}\ \overline{x}]$ is defined as a single continuous domain in the domain of the real numbers. The range of an interval x^I is by definition bounded by a lower bound x_{min} or $\underline{x}$ and an upper bound x_{max} or $\overline{x}$. An interval vector $\{\mathbf{x^I}\}$ or interval matrix $[\mathbf{x^I}]$ has interval scalars for one or more of its components. Representing a certain variability by an interval scalar discards knowledge about the likelihood within the range. Generally this is not good practice, although in a lot of cases probabilistic information is not always present, or even needed as mentioned above. An uncertain variability with known range and unknown likelihood within this range can be modelled consistently with an interval. If the range is also unknown, then a subjective interval has to be chosen. This is also the case for an invariable uncertainty. The subjective choice is limited to a selection of the range in the interval concept. This reduces the likelihood of wrong interpretations compared to the probabilistic concept.

A fuzzy set (L. Zadeh, 1965) is a set in which every member has a degree of membership, represented by the membership function $\mu_{\tilde{x}}(x)$, associated with it. If $\mu_{\tilde{x}}(x) = 1$, x is definitely a member of the fuzzy set. If $\mu_{\tilde{x}}(x) = 0$, x is definitely not a member of the fuzzy set. Considering the membership function as a subjective choice of the analyst, a fuzzy set is most useful to describe uncertainties for which only linguistic information is available. Analysis using fuzzy sets is very often done by using α-cuts (an α-cut contains all the x for which $\mu_{\tilde{x}}(x) > \alpha$ is true), see figure 1. These α-cuts are essentially classical intervals, which means that the interval analysis is the basis of a fuzzy analysis.

Generally an interval finite element (IFE) problem can be represented by (D. Moens, 2008):

$$\{\mathbf{y^s}\} = \{\{\mathbf{y}\} \mid (\{\mathbf{x}\} \in \{\mathbf{x^I}\})(\{\mathbf{y}\} = f(\{\mathbf{x}\}))\} \tag{1}$$

The solution is expressed as a set $\{\mathbf{y^s}\}$ to stress that certain combinations of components of $\{\mathbf{y}\}$ are not necessarily possible. However in most cases the individual ranges of only some components of $\{\mathbf{y}\}$ are really of interest. Essentially two fundamental approaches exist for the solution of this problem: interval arithmetics or optimisation approaches (M. De Munck, 2009a). The former gives large overestimations of the results by neglecting interdependencies of the components of the input $\{\mathbf{x^I}\}$. The latter searches for the minimum and maximum of every component of $\{\mathbf{y}\}$, resulting in the smallest hypercube $\{\mathbf{y^I}\}$ that fits around $\{\mathbf{y^S}\}$. This searching can be done on an approximation

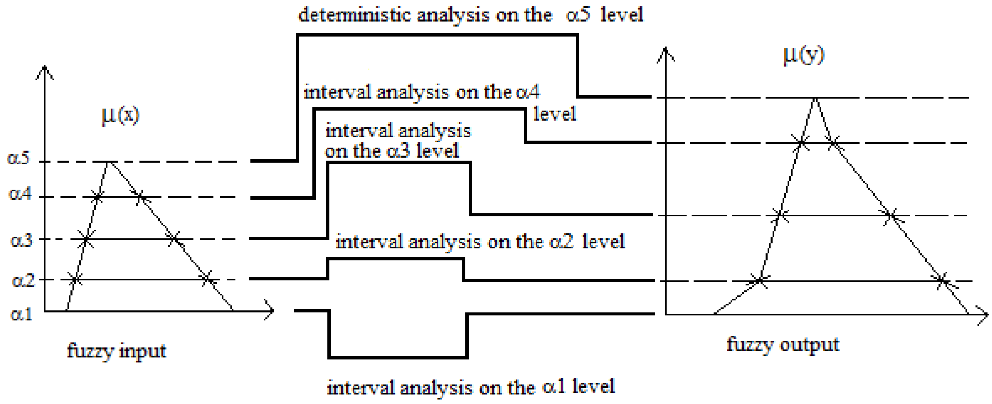

Figure 1. Fuzzy analysis using α-cuts

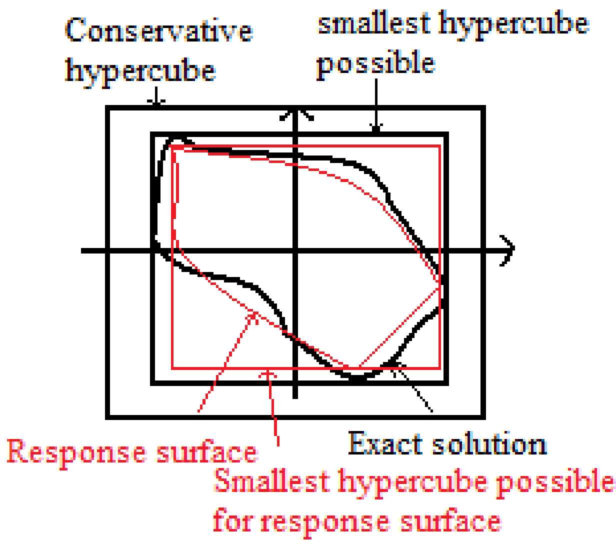

Figure 2. Hypercubic approximations of a two-dimensional output set of an IFE analysis

of the exact solution using response surface techniques (M. De Munck, 2009b). Figure 2 shows the different hypercubic approximations. The interval field technique presented in this article is a response surface technique.

2.2. Fields

A field describes the spatial coherence and distribution of a parameter. In this way a field represents the variation of a physical property inside one realisation of a design, rather than the variation of that property between different realisations of such a design. In a finite element analysis, fields arise on the input side (e.g. pressure distribution) as well as the output side (e.g. deformation). Every component (local value) of such a field is not independent of the other components. In reality a strong correlation exists between the value of a parameter in one place and its value in a nearby place.

To describe spatially correlated variation, numerical modelling often uses shape functions (e.g. the modes used to represent the dynamic behaviour of a structure using the modal superposition technique).

A general static finite element problem is represented by:

$$[\mathbf{K}]\,\{\mathbf{u}\} = \{\mathbf{f}\} \tag{2}$$

with $[\mathbf{K}]$ the stiffness matrix, $\{\mathbf{u}\}$ displacements in the non-constrained degrees of freedom (translations and rotations) and $\{\mathbf{f}\}$ the loads in the non-constrained degrees of freedom. The shape functions $[\mathbf{\Psi}] = [{}_{\mathbf{1}}\mathbf{\Psi} \quad {}_{\mathbf{2}}\mathbf{\Psi} \quad \ldots]$ describe the deformation of a structure by fixing the ratios of the different degrees of freedom in $\{\mathbf{u}\}$. Projecting (2) on these deformation patterns yields:

$$[\mathbf{\Psi}]\,\{\mathbf{w}\} \;=\; \{\mathbf{u}\} \tag{3}$$

$$[\mathbf{\Psi}]^{\,T}\,[\mathbf{K}]\;\,[\mathbf{\Psi}]\,\{\mathbf{w}\} \;=\; [\mathbf{\Psi}]^{\,T}\{\mathbf{f}\} \tag{4}$$

with $\{\mathbf{w}\}$ the weights for the deformation patterns (to find $\{\mathbf{u}\}$).

Each deformation pattern of $[\mathbf{\Psi}]$ represents a possible fixed correlation between the components of the deformation field. It is clear that a realistic set of deformation patterns is mandatory to describe the final results. The selection of such a set of deformation patterns is presented in section 3.1 and will be the basis of the interval field description.

2.3. Non-deterministic fields

A field can depend on a non-deterministic property (e.g. the deformation field of a cantilever beam depends on the stiffness at the boundary).

Using the non-probabilistic concept of an interval vector to describe a non-deterministic field means neglecting the possible correlation between the components of this field. (e.g. figure 3 shows in red an unrealistic solution that an interval vector representation of a deformation field allows to exist) A deformation pattern on the other hand does describe spatial correlation. If the non-determinism can be dealt with in a way it influences the pattern as a whole and not the individual components, interval fields can be used to represent non-deterministic fields. De Gersem (H. De Gersem, 2008) proposes such a technique for dynamic analysis. The general principle for static analysis is explained here.

The uncertainties ($\{\mathbf{x}\} \in \{\mathbf{x}^{\mathbf{I}}\}$) influence the stiffness matrix $[\mathbf{K}(\mathbf{x})]$ and/or the load vector $\{\mathbf{f}(\mathbf{x})\}$. This results in uncertainty on the displacement vector $\{\mathbf{u}(\mathbf{x})\}$. Instead of a direct influence of the uncertainties on the components of $\{\mathbf{u}(\mathbf{x})\}$, deformation patterns $[\mathbf{\Psi}(\mathbf{x})]$ provide an

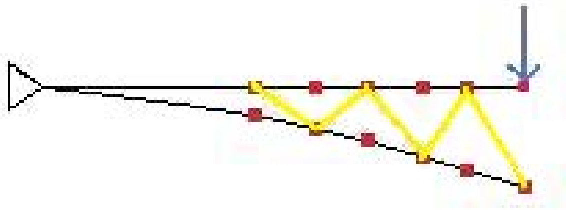

Figure 3. An interval vector represents independent results

intermediate spatial correlation, they will be influenced by the uncertainties. Furthermore, the deformation patterns' dependency on the uncertainties is only approximated $[\tilde{\boldsymbol{\Psi}}(\mathbf{x})]$ based on a few exact evaluations, which results in a response surface technique.

$$[\tilde{\boldsymbol{\Psi}}(\mathbf{x})]\,\{\mathbf{v}\} = \{\tilde{\mathbf{u}}(\mathbf{x})\} \;\tilde{=}\; [\boldsymbol{\Psi}(\mathbf{x})]\,\{\mathbf{w}\} = \{\mathbf{u}(\mathbf{x})\} \tag{5}$$

$$[\tilde{\boldsymbol{\Psi}}(\mathbf{x})]^T\,[\mathbf{K}(\mathbf{x})]\,\,[\tilde{\boldsymbol{\Psi}}(\mathbf{x})]\,\{\mathbf{v}\} = [\tilde{\boldsymbol{\Psi}}(\mathbf{x})]^T\{\mathbf{f}(\mathbf{x})\} \tag{6}$$

The benefits of this technique for interval analysis are twofold:

- Fast approximation of the dependencies using $[\tilde{\boldsymbol{\Psi}}(\mathbf{x})]$
- Smaller system of equations to be solved due to the projection of the original system on a few deformation patterns

This fast implicit link between the input uncertainties and the output field speeds up the calculation of one deterministic analysis at the core of an interval finite element problem. The accuracy of this new procedure is discussed in section 3.2.

3. Interval fields for FE output

The description of the different deformation patterns $[\boldsymbol{\Psi}(\mathbf{x})]$ and the approximation for the way the uncertainties are influencing them $[\tilde{\boldsymbol{\Psi}}(\mathbf{x})]$ are presented first. Secondly, a novel projection technique is presented.

3.1. Deformation patterns for static analysis

For static analysis only a limited number of deformation patterns is commonly used. This section presents briefly three different deformation patterns. To give a mathematical description of these, the degrees of freedom (dofs) of a structure are divided in four sets as shown in table I. The c-set contains all the dofs of which the value is known, the r-set contains all the remaining dofs that are to be constrained to resist rigid body motion. The l-set contains the dofs that are loaded. The remaining dofs are members of the o-set.

Table I. The four subsets of the degrees of freedom (dofs)

c-set	Known, constrained dofs
r-set	Dofs to be constrained to resist rigid body motion
l-set	Dofs that are loaded
o-set	Remaining dofs

The set of rigid body deformation patterns $[\ \mathbf{\Psi^R}\ ]$ contains all the deformation patterns that require no external forces. The calculation is done using

$$[\ \mathbf{\Psi^R}\ ] = \begin{bmatrix} \mathbf{I_{rr}} \\ \mathbf{\Psi^R_{lr}} \\ \mathbf{\Psi^R_{or}} \end{bmatrix} = \begin{bmatrix} \mathbf{I_{rr}} \\ -\begin{bmatrix} \mathbf{K_{ll}} & \mathbf{K_{lo}} \\ \mathbf{K_{ol}} & \mathbf{K_{oo}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{K_{lr}} \\ \mathbf{K_{or}} \end{bmatrix} \end{bmatrix} \tag{7}$$

with a notation like $[\mathbf{K_{rl}}]$ to describe the part of $[\mathbf{K}]$ that shows the influence of the dofs from the l-set on the dofs from the r-set.

The displacement deformation patterns $[\ \mathbf{\Psi^C}\ ]$ are calculated one by one using temporary sets m and p. The m-set (singleton) contains one dof from the l-set, while the p-set contains all the dofs of the o-set and the remaining dofs of the l-set. A displacement deformation pattern is then found by forcing the dof in the m-set to the unit value, while the dofs from the p-set remain unloaded and unconstrained

$$\left\{\ \mathbf{\Psi^C_m}\ \right\} = \left\{ \begin{matrix} \mathbf{I_{mm}} \\ \mathbf{\Psi^C_{pm}} \end{matrix} \right\} = \left\{ \begin{matrix} \mathbf{I_{mm}} \\ -\mathbf{K^{-1}_{pp}K_{pm}} \end{matrix} \right\} \tag{8}$$

The complete set of displacement deformation patterns follows from the iteration of (8) till every dof of the l-set has been the only element in the m-set.

The set of force deformation patterns $[\ \mathbf{\Psi^A}\ ]$ is calculated by successively equalling the load in every dof from the l-set to the unit value

$$[\ \mathbf{\Psi^A}\ ] = \begin{bmatrix} \mathbf{K_{ll}} & \mathbf{K_{lo}} \\ \mathbf{K_{ol}} & \mathbf{K_{oo}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I_{ll}} \\ \mathbf{0} \end{bmatrix} \tag{9}$$

3.2. Approximation for uncertainty dependence and novel projection

An uncertainty $x^I = [x_{min}\ x_{max}]$ changes the system of equations in (7), (8) and (9). The deformation patterns are not calculated exactly for every change in the value of x, instead an approximation based on a nominal and deviatoric pattern is used as in (H. De Gersem, 2008). For a structure without rigid body freedom (no dofs in the r-set) and only one loaded dof (one dof in the l-set):

$$\{\tilde{\mathbf{\Psi}}(\mathbf{x})\} = \{\mathbf{\Psi_{nom}}\} + \{\mathbf{\Psi_{dev}}\}f(\Delta x) \tag{10}$$

with $\Delta x = \frac{x - x_{nom}}{x_{nom}}$ and $f(\Delta x)$ a function with a root for $\Delta x = 0$. Only three exact evaluations of a deformation pattern are used, one for the minimum, maximum and nominal ($x = x_{nom} =$

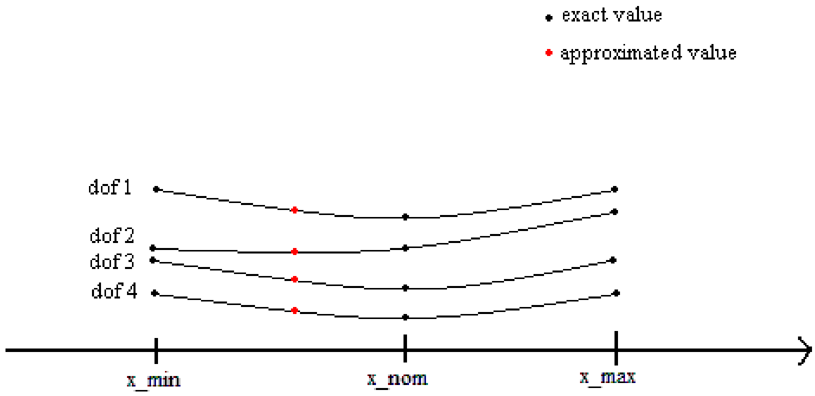

Figure 4. Approximated behaviour of a deformation pattern for an interval x^I

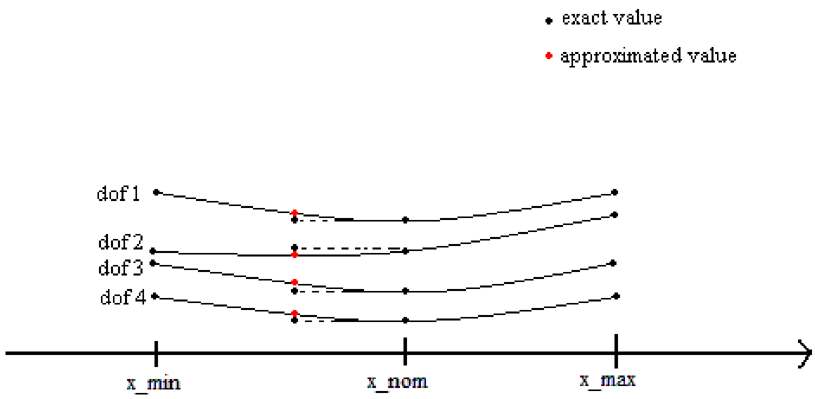

Figure 5. Approximated behaviour of a deformation pattern for an interval x^I with the nominal pattern kept seperated

$\frac{x_{max}+x_{min}}{2}$) value of the uncertainty. Based on these evaluations a quadratic

$$\{\mathbf{\Psi}(\mathbf{x})\}\tilde{=}\{\tilde{\mathbf{\Psi}}(\mathbf{x})\} = \{\mathbf{\Psi_{nom}}\} + \{\mathbf{\Psi_{linear}}\}\Delta x + \{\mathbf{\Psi_{quadratic}}\}\Delta x^2 \tag{11}$$

or exponential

$$\{\mathbf{\Psi}(\mathbf{x})\}\tilde{=}\{\tilde{\mathbf{\Psi}}(\mathbf{x})\} = \{\mathbf{\Psi_{nom}}\} + \{\mathbf{\Psi_{amplitude}}\}(e^{\Delta x} - 1) + \{\mathbf{\Psi_{linear}}\}\Delta x \tag{12}$$

approximation can be constructed. A more advanced approximation can be constructed based on a higher number of exact evaluations.

An approximation for a general influence of an uncertainty x on a deformation pattern for a structure made of four dofs can be illustrated as in figure 4. For a specific value of x the deformation pattern is approximated by the red dots. The solution for the finite element problem will be a simple scaling of this pattern using (5). If in the projection the nominal and deviatoric part of the pattern are kept seperated, rather than immediately adding them together, an extra way of matching the approximated deformation pattern with the exact one becomes available:

$$\{\tilde{\mathbf{\Psi}}(\mathbf{x})\} = \beta\{\mathbf{\Psi_{nom}}\} + \gamma\{\mathbf{\Psi_{dev}}\}f(\Delta x) \tag{13}$$

This equals (10) if $\beta = \gamma = 1$. In the new projection such equality is not mandatory a priori, but instead β and γ are determined by the static finite element problem itself. Figure 5 shows the nominal and the deviatoric (the difference between the red and the black dots for a specific value of x) pattern for the new projection:

$$\left[\mathbf{\Psi_{nom}}\ \ \mathbf{\Psi_{dev}}f(\Delta x)\right]^T [\mathbf{K}(\mathbf{x})]\ \left[\mathbf{\Psi_{nom}}\ \ \mathbf{\Psi_{dev}}f(\Delta x)\right]\{\mathbf{v}\} = \left[\mathbf{\Psi_{nom}}\ \ \mathbf{\Psi_{dev}}(f\Delta x)\right]^T\{\mathbf{f}(\mathbf{x})\} \tag{14}$$

$$\left[\mathbf{\Psi_{nom}}\ \ \mathbf{\Psi_{dev}}f(\Delta x)\right]\{\mathbf{v}\} = \{\tilde{\mathbf{u}}\}\tilde{=}\{\mathbf{u}\} \tag{15}$$

Extending this new projection for a deformation pattern influenced by more than one uncertainty means using (6) with:

$$[\tilde{\mathbf{\Psi}}(\mathbf{x})] = [\mathbf{\Psi_{nom}}\ \ \mathbf{\Psi^1_{dev}}f(\Delta x^1)\ \ \mathbf{\Psi^2_{dev}}f(\Delta x^2)\ \ldots] \tag{16}$$

Furthermore for a structure loaded in different dofs, the l-set consists of more than one degree of freedom. If such a structure is influenced by more than one uncertainty the projection is done with:

$$[\tilde{\mathbf{\Psi}}(\mathbf{x})] = \begin{matrix} [{}_\mathbf{1}\mathbf{\Psi_{nom}} & {}_\mathbf{1}\mathbf{\Psi^1_{dev}}f(\Delta x^1) & {}_\mathbf{1}\mathbf{\Psi^2_{dev}}f(\Delta x^2) & \ldots \\ {}_\mathbf{2}\mathbf{\Psi_{nom}} & {}_\mathbf{2}\mathbf{\Psi^1_{dev}}f(\Delta x^1) & {}_\mathbf{2}\mathbf{\Psi^2_{dev}}f(\Delta x^2) & \ldots \\ \ldots & \ldots & \ldots & \ldots] \end{matrix} \tag{17}$$

The extra calculation cost for this novel projection is limited since the nominal and deviatoric components are calculated anyway. The only extra work to be done is solving a system of equations (after projection) that is slightly larger (one extra unknown weighting for every uncertainty).

4. Example

In the following numerical case study, the interval field representation accuracy is checked. The static analysis of a subpart of the small launcher VEGA (*ESA*) (see figure 6) is done using an interval field representation and the classic solution method, the difference between both (the error of the response surface) is presented. A comparison of the error made using the classic projection (adding the nominal and all the deviatoric parts) and the novel projection (16) is done.

The original FE model is coarsened to a reduced model with 6726 DOF's. The structure is subject to a vertical nodal force, with magnitude $-10kN$, and it is clamped at the lower side. The effect of 5 different sizing uncertain parameters (see table II) on the static response of the whole structure is investigated. The parameter intervals are equidistantly sampled in 5 points. Every possible combination of the uncertainties is calculated, resulting in $5^5 = 3125$ cases. The error of every case is calculated using:

$$error = \frac{\sum_{n=1}^{n_{DOF}} \mid u_n - \tilde{u}_n \mid}{n_{DOF}u_{max}} \tag{18}$$

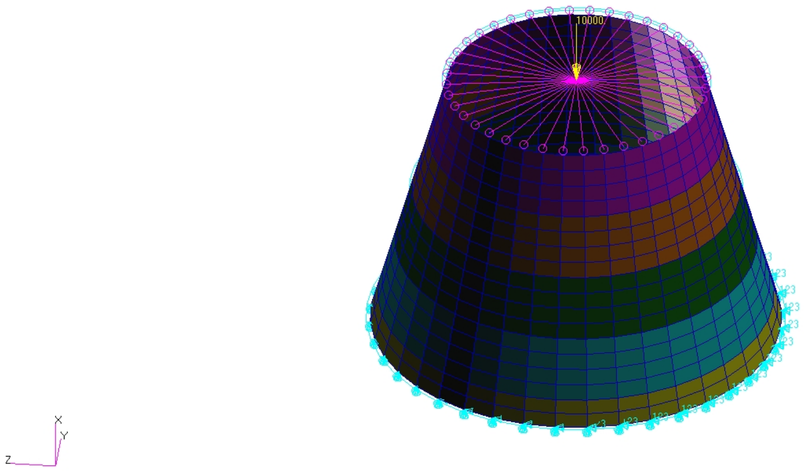

Figure 6. The FE model of the subpart of the small launcher VEGA, clearly showing the five different shell properties, the load and boundary conditions

Table II. The uncertainties on the thicknesses

Notation	Min $[mm]$	Nominal $[mm]$	Max $[mm]$
t_1	3	4	5
t_2	4	5	6
t_3	4	6	8
t_4	4	6	8
t_5	4	7	10

with u_{max} the maximum value of all the dofs in the structure for the actual value of the uncertainties and n_{DOF} the number of dofs (here equal to 6726).

Figure 7 presents the results. Comparing the mean error over all the cases. The classic projection results in an error (with mean 0.0013 and standard deviation 0.0007) twice as large as the error (with mean 0.00055 and standard deviation 0.0003) resulting from the novel projection with (16).

The solution of a static finite element problem using the interval field description presented here is sufficiently accurate. The novel projection technique results in a lower and less fluctuating error.

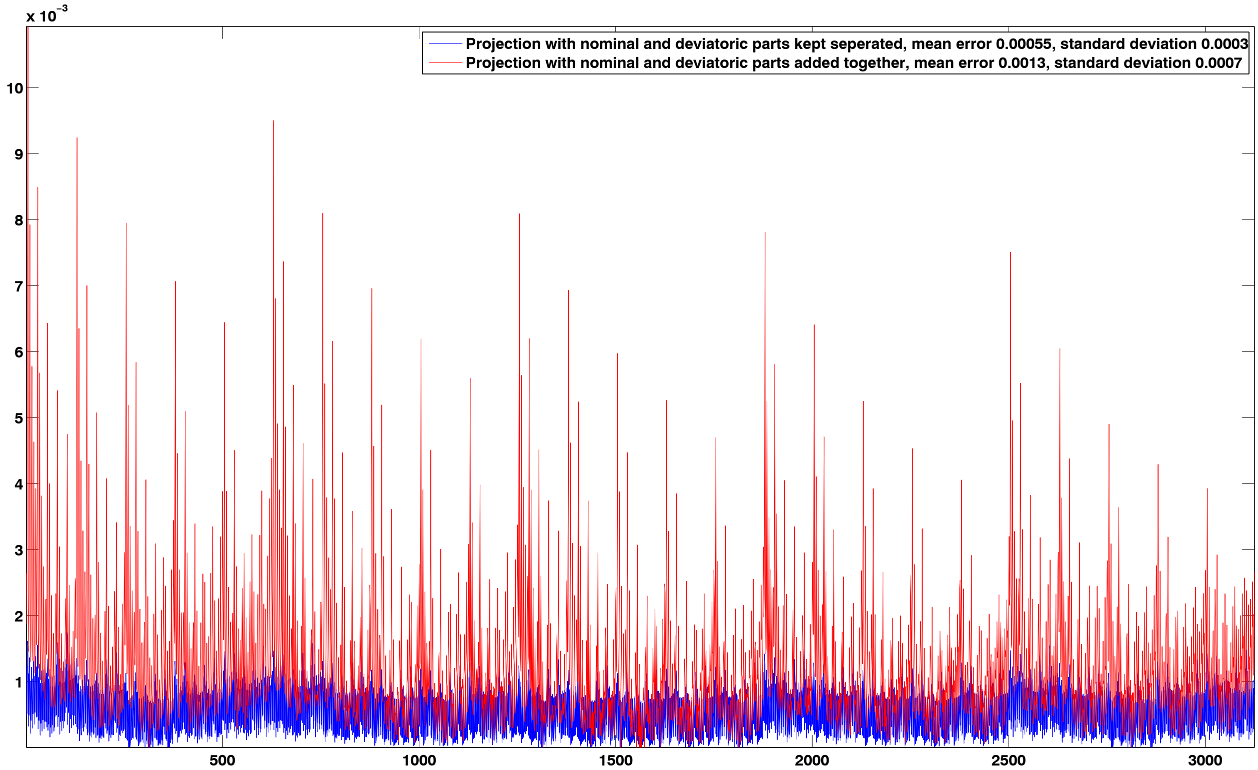

Figure 7. Error over all the 3125 cases calculated with (18), in blue for projection with (16) and in red for the classic projection (adding the nominal and all the deviatoric parts)

5. Conclusions

Using shape functions allows for the development of an interval field representation of uncertainties. This representation provides a fast implicit link between the input uncertainties and the output field for any possible interval technique applied in the uncertainty space on a reduced model.

This non-probabilistic concept is elaborated for the solution of a static finite element problem. Apart from the advantage of the spatial correlation being represented in a coherent manner, the output space is reduced as well. A novel projection is proposed and illustrated with an example. Based on the results of this example, the method proves to be sufficiently accurate over the complete uncertainty space.

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