Primary and Derived Variables with the Same Accuracy in Interval Finite Elements

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Abstract: This paper addresses the main challenge in interval computations which is to minimize the overestimation in the target quantities. When sharp enclosures for the primary variables are achievable in a given formulation such as the displacements in Interval Finite Elements (IFEM) the calculated enclosures for secondary or derived quantities such as stresses usually obtained with significantly increased overestimation. One should follow special treatment in order to decrease the overestimation in the derived quantities see Muhanna, Zhang, and Mullen (2007), Neumaier and Pownuk (2007). In this work we introduce a new formulation for Interval Finite Element Methods where both primary and derived quantities of interest are included in the original uncertain system as primary variables. The formulation is based on the variational approach and Lagrange multiplier method by imposing certain constraints that allows the Lagrange multipliers them-selves to be the derived quantities. Numerical results of this new formulation are illustrated in a number of example problems.

Keywords: Interval; Uncertainty; Dependent Variables; Finite Elements.

1. Introduction

Since the early development of Interval Finite Element Methods (IFEM) during the mid nineties of last century (Koyluoglu, H. U., Cakmak, A. S., and Nielson, S. R. K. 1995, Muhanna, R. L. and Mullen, R. L. 1995, Nakagiri S. and Yoshikawa, N. 1996, Rao, S. S. and Sawyer, P. 1995, Rao, S. S. and Berke, L. 1997, Rao, S.S., and Chen Li 1998.) researchers have focused among other issues on two major problems; the first is how to obtain solutions for the resulting linear interval system of equations with reasonable bounds on the system response that make sense from practical point of view, or in other words with the least possible overestimation of their bounding intervals, the second is how to obtain reasonable bounds on the derived quantities that are functions of the system response. For example, when the system response is the displacement, the derived quantities might be forces or stresses which are functions of the displacements. Obtaining tight bounds on the derived quantities has been a tougher challenge due to the existing

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dependency of these quantities on the primary dependent variables which are already overestimated. So far, the derived quantities are obtained with significantly increased overestimation.

A significant effort has been made in the work of Zhang (2005) to control the additional overestimation in the values of the derived quantities; the derived quantities have been calculated by an implicit substitution of the primary quantities. In addition to calculating rigorous bounds on the solution of the resulting linear interval system, a special treatment has been developed to handle the overestimation in the derived quantities. Instead of first evaluating the primary quantities and then substituting the obtained values in the expression for the derived quantities, the expression for the primary quantities has been substituted before its evaluation in the derived quantities expression and both were evaluated simultaneously preventing a large amount of overestimation in the values of derived quantities. In spite of the advancement provided by this approach, still it is conditioned by the original IFEM formulation and the special treatment of required transformations.

A significant improvement in the formulation of IFEM with application to truss problems has been introduced in the work of Neumaier and Pownuk (2007). This work has presented an iterative method for computing rigorous bounds on the solution of linear interval systems, with a computable overestimation factor that is frequently quite small. This approach has been demonstrated by solving truss problems with over 5000 variables and over 10000 interval parameters, with excellent bounds for up to about 10% input uncertainty. Although, no calculated derived quantities have been reported in this work, a formulation has been introduced for the calculation of derived quantities by intersecting the simple enclosure $\mathbf{z} = Z(\mathbf{u})$, where $\mathbf{z}$ depends linearly or nonlinearly on the solution $\mathbf{u}$ of the uncertain system with another enclosure obtained from the centered form (Neumaier and Pownuk, 2007, Eq. 4.13, pp 157). In spite of the provided improvement in this formulation, the two-step approach will result in additional overestimation when evaluating the derived quantities.

It is quite clear that among other factors, the issue of obtaining tight enclosures for the primary variables as well as for the derived quantities is conditioned by IFEM formulation and the methods used for the evaluation of the derived quantities. In this work we introduce a new mixed formulation for Interval Finite Element Methods where the derived quantities of the conventional formulation are treated as dependent variables along with the primary variables. The formulation uses the mixed variational approach based on the Lagrange multiplier method. The system solution provides the primary variables along with the Lagrange multipliers which represent the derived quantities themselves. Numerical results of this new formulation are illustrated in a number of example problems.

2. Formulation

In the current formulation, our focus will be on two major issues:

1. Obtaining the secondary variables (derived) such as forces and stresses of the conventional displacement FEM along with the primary variables (displacements) and with the same accuracy of the primary ones.

2. Reducing of overestimation in the bounds on the system response due to the coupling and transformation in the conventional FEM formulation as well as due to the nature of used interval linear solvers (Muhanna and Mullen, 2001).

We will begin the formulation with a short theoretical background with the hope that it will facilitate a clearer understanding of the procedure followed in the present formulation. Interval quantities will be introduced in boldface non-italic font.

2.1. SECONDARY VARIABLES

Mixed or hybrid variational formulations are those where secondary variables of the conventional formulation are treated as dependent variables along with the primary variables. Most often these formulations are developed with the objective of determining the secondary variables, which are often quantities of practical interest, directly rather than from post-computations. Mixed formulations are based on stationary principles. A stationary principle is one in which the functional attains neither a minimum nor a maximum in its argument. In fact, a functional can attain a maximum with respect to one set of variables and a minimum with respect to another set of variables involved in the functional. An example of such functionals is provided by the functional based on the Lagrange multiplier method (Reddy, 2002). The Lagrange multiplier method, which forms the basis for the present mixed formulation method, will be introduced briefly in the next section.

2.2. LAGRANGE MULTIPLIER METHOD

The Lagrange multiplier method is one in which the minimum of a functional with linear equality constraints is determined. If we consider the problem of finding the minimum of a functional $I(u, v)$,

$$I(u,v) = \int_a^b F(x,u,u',v,v')dx \tag{1}$$

subjected to the constraint

$$G(u,u',v,v') = 0 \tag{2}$$

where u, v, u' and v' are the dependent variables and their first derivatives, respectively. The necessary condition for the minimum of $I(u,v)$ is $\delta I = 0$. We have

$$0 = \delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta u' \right) dx \tag{3}$$

Since u and v must satisfy the constraint condition given by Eq. (2), the variation δu and δv are related by:

$$0 = \delta G = \int_a^b \left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta u' \right) dx \tag{4}$$

The Lagrange multiplier method consists of multiplying Eq. (4) with an arbitrary parameter λ, integrating over the interval (a, b), and adding the results to Eq. (3). The multiplier λ is called the Lagrange multiplier. The Lagrange's method can be viewed as one of determining u, v and λ by setting the first variation of the *modified* functional

$$L(u,v,\lambda) \equiv I(u,v) + \int_a^b \lambda G(u,u',v,v')dx = \int_a^b (F+\lambda G)dx \tag{5}$$

to zero.

We have

$$\left.\begin{aligned}
0 = \delta L &= \int_a^b (F+\lambda G)dx \\
&= \int_a^b (\delta F + \delta\lambda G + \lambda \delta G)dx \\
&= \int_a^b \left[\left(\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u}\right)\delta u + \left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'}\right)\delta u' + \left(\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v}\right)\delta v + \left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'}\right)\delta v' + \delta\lambda G\right]dx \\
&= \int_a^b \left\{\left[\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'}\right)\right]\delta u + \left[\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'}\right)\right]\delta v + G\delta\lambda\right\}dx
\end{aligned}\right\} \tag{6}$$

The boundary terms vanish because $\delta u(a)$, $\delta u(b)$, $\delta v(a)$, $\delta v(b)$ = 0. Suppose that δu is independent and δv is related to δu by Eq. (4). We choose λ such that the coefficient of δv is zero. Then by the fundamental lemma of variational calculus, it follows that (because δu is arbitrary) the coefficient of δu is also zero. Thus we have:

$$\left.\begin{aligned}
&\frac{\partial}{\partial u}(F+\lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial u'}(F+\lambda G)\right] = 0 \\
&\frac{\partial}{\partial v}(F+\lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial v'}(F+\lambda G)\right] = 0 \\
&G(u,u',v,v') = 0
\end{aligned}\right\} \tag{7}$$

Equations (7) are the Euler equations of the functional $L(u,v,\lambda) \equiv \int_a^b (F+\lambda G)dx$ from which the dependent variables u, v, and λ can be determined at the same time. In general, one can introduce the secondary variable $z = Z(u)$ as the constraint $G = \{[z\text{-}Z(u)]^2\} = 0$. Alternatively, we have found that the Lagrange multiplier λ is the pursued secondary variable by judicious choice of the constraint condition $G(u,$

$u', v, v')$ = 0. In the next section we will illustrate the use of the Lagrange multiplier method in discrete structural models.

2.3. DISCRETE STRUCTURAL MODELS

In steady-state analysis, the variational formulation for a discrete structural model within the context of Finite Element Method (FEM) is given in the following form of the total potential energy functional (Gallagher 1975, Bathe 1996)

$$\Pi = \frac{1}{2}U^T KU - U^T P \tag{8}$$

with the conditions

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad for\ all\ \ i \tag{9}$$

where Π, K, U, and P are total potential energy, stiffness matrix, displacement vector, and load vector respectively. Assume that we want to impose onto the solution the m linearly independent discrete constraints $CU = V$ where C is a matrix of order $m \times n$. In the Lagrange multiplier method we amend the right-hand side of Eq. (8) to obtain

$$\Pi^* = \frac{1}{2}U^T KU - U^T P + \lambda^T (CU - V) \tag{10}$$

where λ is a vector of m Lagrange multipliers. Invoking the stationarity of Π^*, that is $\delta\Pi^*$=0, we obtain

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} P \\ V \end{pmatrix} \tag{11}$$

The solution of Eq. (11) will provide the values of dependent variable U and λ at the same time.

The present interval formulation, which will be presented in the next section, is based on the Element-By-Element (EBE) finite element technique developed in the work of Muhanna and Mullen 2001. In the EBE method, each element has its own set of nodes, but the set of elements is disassembled, so that a node belongs to a single element. A set of additional constraints is introduced to force unknowns associated with coincident nodes to have identical values. Thus, the constraint equation CU=V takes the form

$$CU = 0 \tag{12}$$

where C is the constraint matrix, and equation (11) takes the form:

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix}\begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix} \tag{13}$$

2.4. Present Interval Formulation

The main sources of overestimation in the formulation of IFEM are the multiple occurrences of the same interval variable (*dependency problem*), the width of interval quantities, the problem size, and the problem complexity, in addition to the nature of the used interval solver of the interval linear system of equations. While the present formulation is valid for the FEM models in solid and structural mechanics problems, the truss model will be used here to illustrate the applicability and efficiency of the present formulation of without any loss of generality.

To illustrate the present formulation, let us consider a typical two dimensional truss bar finite element as shown in Figure 1. According to finite element formulation (Bathe, 1996, Gallagher, 1995, Zienkiewicz and Taylor, 2000) the global finite element model of a truss system is given in the following form:

$$KU = P \tag{14}$$

where K is the assembled global stiffness matrix, P is the global load vector, and U is the unknown global displacement vector.

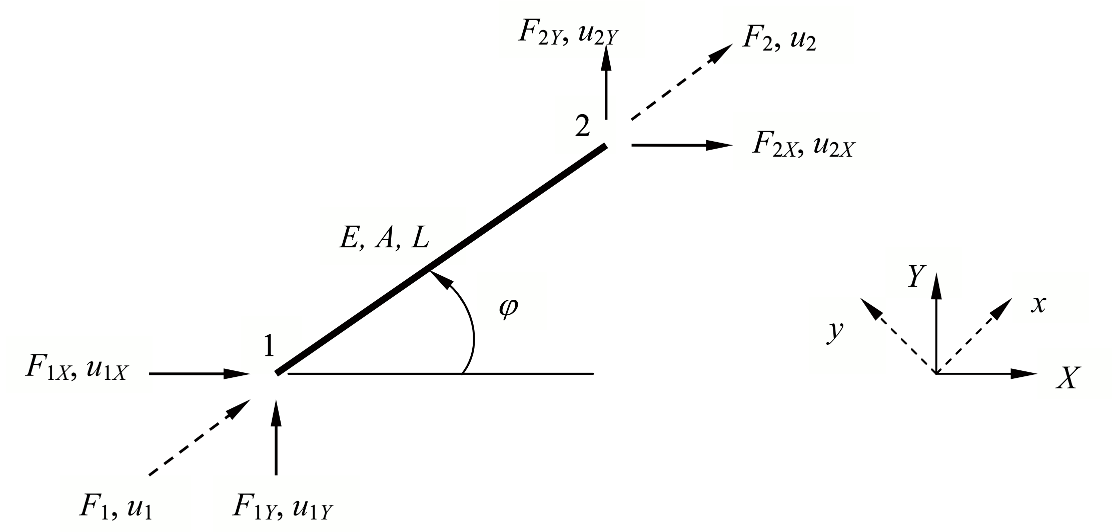

Figure 1. A typical truss bar element, local and global coordinates.

Using boldface non-italic font for interval quantities, the interval form of Eq. (14) will be

$$\mathbf{KU} = \mathbf{P} \tag{15}$$

where **K**, **U**, and **P** are the interval global stiffness matrix, interval global displacement vector, and interval global load vector, respectively. The interval solution of Eq. (15) results in a significant overestimation in the system response; a comprehensive discussion can be found in (Muhanna and Mullen 2001). In addition, internal forces and stresses are quantities of practical interest in design. Usually interval element forces can be obtained as:

$$\mathbf{F_e} = \mathbf{k_e} L_\mathbf{e} \mathbf{U} \tag{16}$$

where $\mathbf{F}_e$, $\mathbf{k}_e$, L_e are global interval vector of element forces, global interval element stiffness matrix, and element Boolean matrix, respectively. Once again, an additional overestimation in the values of forces is obtained due to the dependency between **U** and $\mathbf{k}_e$. Frequently, element forces are pursued in local coordinate system that will require the transformation from the global coordinates to the local ones in the form:

$$\mathbf{F}_{e,local} = T_e\, \mathbf{k}_e L_e \mathbf{U} \tag{17}$$

where $\mathbf{F}_{e,\ local}$ and T_e are the local vector of interval element forces and the corresponding transformation matrix, respectively. The transformation procedure will provide an additional overestimation.

The current formulation is attempting to reduce overestimation due to coupling in the FEM assembling process, multiple occurrences of interval quantities, transformation, and solving the final system of interval linear equations. In addition this formulation will introduce the derived quantities such as forces and stresses as dependent variables which will be obtained along with displacements when the system is solved.

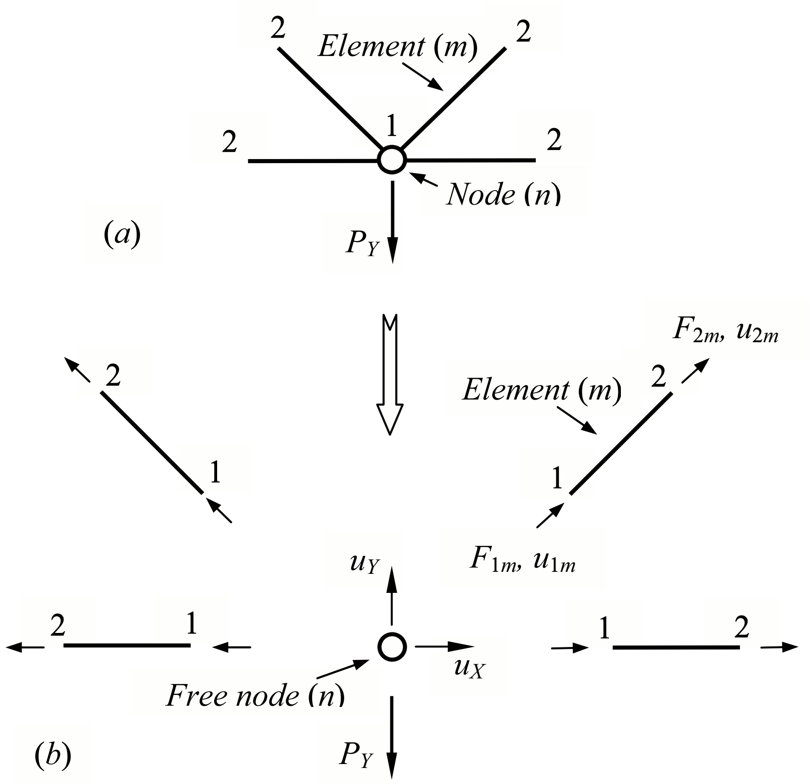

Figure 2. A typical node of a truss problem. (*a*) Conventional formulation. (*b*) Present formulation.

In the conventional formulation of FEM Figure 2 (a), after deriving the local elements' stiffness matrices along with the local elements' load vectors the system will be transformed to the global system and assembled based on compatibility requirements resulting in the equilibrium system given by Eq. (14). In the present formulation the following steps are followed:

1. Considering a typical node of the truss system Figure 2 (a), elements and nodes are disassembled as in Figure 2 (b). The typical node is called a *free node* and is given along with all pertinent variables in the global coordinate system. Displacements are u_X and u_Y and applied forces are P_X and P_Y. The free node displacements are considered as independent of those of the elements.
2. All coinciding elements at the free node along with pertinent variables are given in local coordinate system. For example, element m has the end nodes 1 and 2, the local displacements u_{1m} and u_{2m}, and the local forces F_{1m} and F_{2m}. By doing that, each element is treated as having independent degrees of freedom in its own local coordinate system.
3. The system will be assembled imposing the discrete constraints C_{mi} to ensure the equality between the free node displacements and those of the elements. Where i is the number of constraints per element.

This procedure will result in the following system of equations:

$$\begin{pmatrix} \mathbf{k} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ 0 \end{pmatrix} \tag{18}$$

where $\mathbf{k}$ is an interval matrix consists of the individual elements' local stiffness and zeros at the bottom corresponding the free nodes' degrees of freedom and have the following structure:

$$\mathbf{k} = \begin{pmatrix} \mathbf{k}_1 & -\mathbf{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{k}_1 & \mathbf{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}_n & -\mathbf{k}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{k}_n & \mathbf{k}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{1X} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1Y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{mX} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{mY} \end{pmatrix} \tag{19}$$

and

$$\mathbf{k}_i = \frac{\mathbf{E_i A}_i}{L_i} \tag{20}$$

where $\mathbf{E}_i$, $\mathbf{A}_i$, and L_i are the interval modulus of elasticity, the interval cross-sectional area, and the length of each element, respectively.
Matrix C has the dimensions ($k \times l$), where k = number of elements' degrees of freedom ($2 \times$ number of elements in the truss bar element case), and l = total number of the system's degrees of freedom. The entries of the matrix are equality constraints of the following type

$$\mathbf{u}_{1i} + \mathbf{u}_{jX} cos\varphi_i + \mathbf{u}_{jY} sin\varphi_i = 0 \tag{21}$$

Where $\mathbf{u}_{1i}$ is the local displacement of the node 1 that belongs to i^{th} element, $\mathbf{u}_{jX}$ and $\mathbf{u}_{jY}$ are the X and Y global displacements of j^{th} free node coinciding with the 1st node of the i^{th} element. Elements of C^T are shown in Eq. (22)

$\mathbf{U}$ is a vector of size $s \times 1$ where s is the number of elements' local degrees of freedom + number of free nodes' global degrees of freedom. The entries of the vector are the interval local displacements of elements followed by interval global displacements of the *free nodes* as shown in Eq. (23).Vector $\boldsymbol{\lambda}$ has the dimension of total number of elements' local degrees of freedom and given in Eq. (24). The entries of the vector are the interval Lagrange multipliers that represent minus the local element forces in this case.

$$C^T = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cos\varphi_1 & 0 & \cdots \\ \sin\varphi_1 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & \cos\varphi_1 & \cdots \\ 0 & \sin\varphi_1 & \cdots \end{pmatrix} \tag{22} \qquad \mathbf{U} = \begin{pmatrix} \mathbf{u}_{11} \\ \mathbf{u}_{21} \\ \vdots \\ \mathbf{u}_{1n} \\ \mathbf{u}_{2n} \\ \mathbf{u}_{1X} \\ \mathbf{u}_{1Y} \\ \vdots \\ \mathbf{u}_{mX} \\ \mathbf{u}_{mY} \end{pmatrix} \quad (23) \qquad \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_{11} \\ \boldsymbol{\lambda}_{21} \\ \vdots \\ \boldsymbol{\lambda}_{1n} \\ \boldsymbol{\lambda}_{2n} \end{pmatrix} \quad (24)$$

Vector $\mathbf{P}$ is the interval load vector and has the dimension equal to the sum of elements degrees of freedom and the free nodes degrees of freedom. The entries of the vector are given in Eq. (25).

$$\mathbf{P}^T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \mathbf{P}_{1X} & \mathbf{P}_{1Y} & \cdots & \mathbf{P}_{mX} & \mathbf{P}_{mY} \end{pmatrix} \tag{25}$$

The accuracy of the system solution depends mainly on the structure of Eq. (18) and on the nature of the used solver. The solution of the interval system (18) provides the enclosures of the values of dependent variables which are the interval displacements **U** and interval element forces $\boldsymbol{\lambda}$. An iterative solver is discussed in the next section.

2.5. ITERATIVE ENCLOSURES

The best known method for obtaining very sharp enclosures of interval linear system of equations that have the structure introduced in Eq. (26) is the iterative method developed in the work of Neumaier and Pownuk, (2007). The current formulation results in the interval linear system of equations given in (18) which can be introduced in the same structure of the following system:

$$(K + B\,\mathbf{D}\,A)\mathbf{u} = a + F\,\mathbf{b} \tag{26}$$

with interval quantities in **D** and **b** only. The quality of the enclosures is known to be superior only in the special case where **D** is diagonal, however It is expected that the enclosures are also good in case **D** is block diagonal with diagonal blocks small compared to the matrix size. The solution is obtained by performing the following iterative scheme:

$$\mathbf{v} = \{ACa) + (ACF)\mathbf{b} + (ACB)\mathbf{d}\} \cap \mathbf{v}, \quad \mathbf{d} = \{(D_0 - \mathbf{D})\mathbf{v} \cap \mathbf{d} \tag{27}$$

until some stopping criteria, and then the following enclosure is obtained:

$$\mathbf{u} = (Ca) + (CF)\mathbf{b} + (CB)\mathbf{d} \tag{28}$$

Where

$$\begin{aligned} &C := (K + BD_0A)^{-1} \\ &\mathbf{u} = Ca + CF\mathbf{b} + CB\mathbf{d} \\ &\mathbf{v} = ACa + ACF\mathbf{b} + ACB\mathbf{d} \\ &\mathbf{d} = (D_0 - \mathbf{D})\mathbf{v} \end{aligned} \tag{29}$$

In Neumaier's work only excellent enclosures of the interval displacements are obtained. However for the derived quantities such as forces he suggested an improved enclosure by intersecting the simple enclosure $\mathbf{z} = Z(\mathbf{u})$ with the following enclosure:

$$\mathbf{z} = Z(CF\,mid\,\mathbf{b}) + (\mathbf{S}CF)(\mathbf{b} - mid\,\mathbf{b}) + (\mathbf{S}CB)\mathbf{d} \tag{28}$$

which results in additional significant overestimation
The current formulation allows obtaining the interval displacement **U** and the accompanied interval derived quantities $\boldsymbol{\lambda}$ with the same accuracy. A number of examples are introduced in the following sections that illustrate the excellent accuracy of the developed method.

3. Example Problems

Three example problems are chosen to illustrate the present approach and also to demonstrate its ability to obtain sharp bounds to the displacements and forces even in the presence of large uncertainties and large number of interval variables.

The first example chosen is an eleven bar truss (Muhanna and Mullen, 2001) as shown in Figure 3. The displacements and forces of this statically indeterminate truss are dependent on the uncertainty present in the modulus of elasticity and load. The results of this example allow us to investigate the effect of load and stiffness uncertainty on the displacements and forces using various approaches presented. A cantilever truss as shown in Figure 8 is chosen as second example. This truss structure is a benchmark problem adopted from the website of the Center for Reliable Computing (http://www.gtsav.gatech.edu/rec/benchmarks.html). The objective of choosing this example is to demonstrate the applicability, computational efficiency and scalability of the present approach for structures with complex configuration with a large number of interval parameters. The third example problem is a fifteen bar truss as shown in Figure 11 (Zhang, 2005). This truss is internally indeterminate but externally determinate. Thus, the support reactions as well as axial forces in elements 1,2,14 and 15 are independent of structural stiffness although structural stiffness is uncertain while the axial forces in the remaining elements are dependent on structural stiffness. The objective of choosing this example is to verify the ability of Neumaier's approach and present approach to capture this phenomenon.

Considering the first example problem, the eleven bar truss is subjected to a concentrated load of 15 kN, applied at the middle lower joint. The deterministic value of Young's modulus of each element is $E_i=2\times10^{11}$N/m^2, i=1,2,…,11, while the cross sectional area is 0.01m^2. The modulus of elasticity of each element is assumed to vary independently. The solution is computed using four approaches viz. a combinatorial approach, Krawczyk Fixed Point Iteration (FPI)-this approach uses the same system structure as given by Neumaier and a fixed point iterative solver, Neumaier's approach and the present approach.

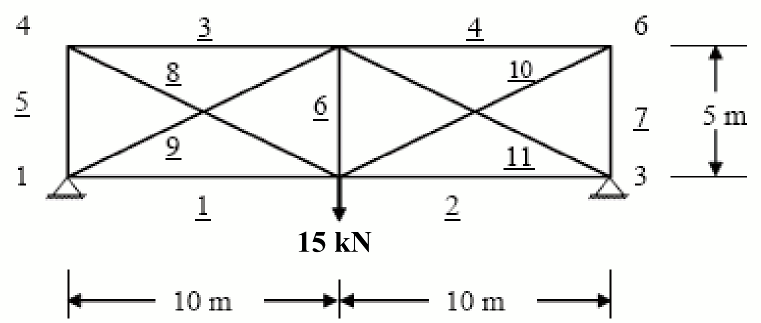

Figure 3. Eleven bar truss.

Tables 1, 2 and 3 show the computed values of selected displacements (vertical displacement V_2 at node 2, horizontal displacement U_4 and vertical displacement V_4 at node 4) using four approaches mentioned above for uncertainties of 1 percent, 12 percent and 25 percent (±0.5%, ±6% and ±12.5% from the mean value of

E, respectively. The error in width is computed as $100 \times \left(\dfrac{width\,of\ computed\ solution}{width\,of\ combinatorial\ solution} - 1 \right)$. It is observed from Table 1 that the displacements obtained using the present approach at 1 percent uncertainty provides sharp enclosure to the displacements obtained using the combinatorial approach and also agree very well with the displacements obtained using Neumaier's approach and Krawczyk FPI . *It is observed from Table* 2 *that for a large uncertainty of* 12% *(and beyond), Krawczyk FPI fails to provide any solution (no enclosure is reached).* However the present approach and Neumaier's approach still provide solution with reasonable sharpness for this level of uncertainty. For a comparison, the error in width of vertical displacement at node 2 (V_2) varies from 0.71 at 1% uncertainty to 9.02 at 12% uncertainty, while the error in width is 20.24 at 25% uncertainty. Thus it is observed that both Neumaier's approach and the present approach provide guaranteed bounds on the combinatorial approach.

Table 1 Eleven bar truss - displacements for 1% uncertainty in the modulus of elasticity (E)

	$V_2 \times 10^{-5}$		$U_4 \times 10^{-5}$		$V_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-15.024443	-14.874946	2.923326	3.003075	-0.748810	-0.732849
Krawczyk FPI	-15.024603	-14.874039	2.922944	3.003347	-0.748841	-0.732732
Error %(width)	0.71		0.81		0.93	
Neumaier's approach	-15.024602	-14.874039	2.922943	3.003347	-0.748841	-0.732731
Error %(width)	0.71		0.81		0.93	
Present approach	-15.024603	-14.874039	2.922944	3.003347	-0.748841	-0.732732
Error %(width)	0.71		0.81		0.93	

Table 2 Eleven bar truss -displacements for 12% uncertainty in the modulus of elasticity (E)

	$V_2 \times 10^{-5}$		$U_4 \times 10^{-5}$		$V_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-15.903532	-14.103133	2.490376	3.451843	-0.843182	-0.650879
Krawczyk FPI	---	---	---	---	---	---
Neumaier's approach	-15.930764	-13.967877	2.431895	3.4943960	-0.848475	-0.633096
Error %(width)	9.02		10.50		11.99	
Present approach	-15.930764	-13.967877	2.431895	3.494396	-0.848475	-0.633096
Error %(width)	9.02		10.50		11.99	

Table 3 Eleven bar truss -displacements for 25% uncertainty in the modulus of elasticity (E)

	$V_2 \times 10^{-5}$		$U_4 \times 10^{-5}$		$V_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-17.084938	-13.288285	1.983562	4.013033	-0.971261	-0.565208
Neumaier's approach	-17.231940	-12.666701	1.705818	4.220473	-0.999561	-0.482011
Error %(width)	20.24		23.90		27.45	
Present approach	-17.231940	-12.666701	1.705818	4.220473	-0.999561	-0.482011
Error %(width)	20.24		23.90		27.45	

Figure 4 shows the computed interval values of vertical displacement V_2 at node 2. The figure depicts the variation of the width of the present approach and the combinatorial approach with the variation of modulus of Elasticity (E) from its mean value. It is observed from this figure that the present solution encloses the combinatorial solution at all values of variation from 0 percent to 25 percent. A similar behaviour is observed in the plot for variation of width of axial force N_3 in element 3 in Figure 5.

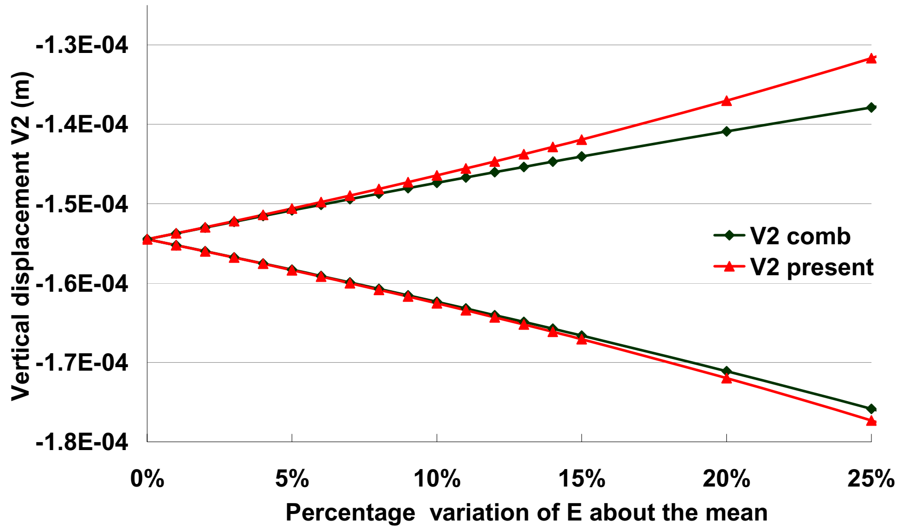

Figure 4. Eleven bar truss - variation of vertical displacement at node 2 with uncertainty of E. Comb. solution vs. present solution.

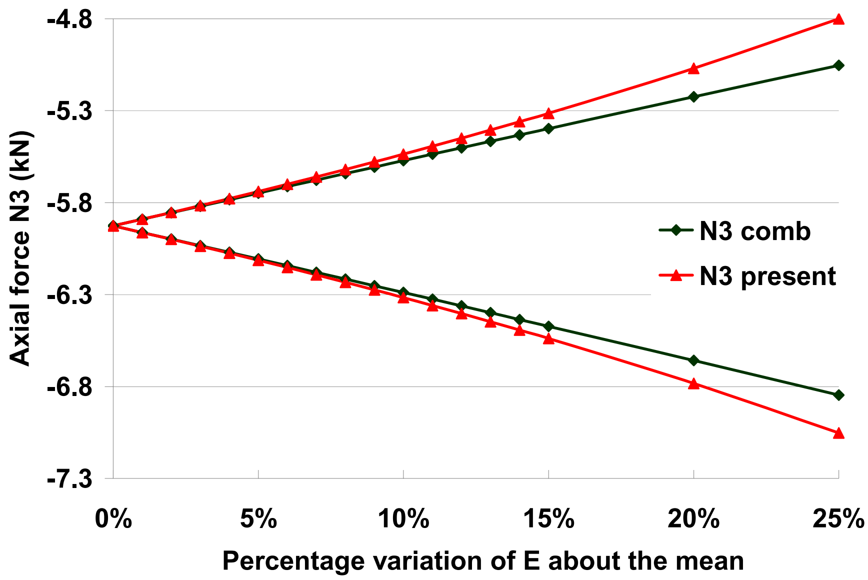

Figure 5. Eleven bar truss – variation of axial force in element 3 with uncertainty of E. Comb. solution vs. present solution.

Axial forces are computed for the eleven bar truss using the following approaches

a) Simple enclosure $\mathbf{z}_1(\mathbf{u})$

b) $\mathbf{z}_2(\mathbf{u})$, the intersection of $\mathbf{z}_1(\mathbf{u})$ with the enclosure obtained using Eq. (28) of Neumaier.

c) Present approach

The interval values of axial forces in elements 3 and 9 (N_3 and N_9) are presented in Table 4. It is clearly observed from this table that the present method provides very sharp enclosure to the forces in comparison with both the enclosures suggested by Neumaier, even at an uncertainty as large as 10%. This illustrates the ability of the present approach to obtain sharp bounds to displacements and forces even at larger values of uncertainty.

The eleven bar truss mentioned above is analysed once again at various levels of uncertainty of Young's modulus and load and the results are tabulated. Table 5 presents the computed values of selected displacements for the second case study with uncertainty of Young's modulus and load being 1 percent. Table 6 presents the corresponding values of axial forces in elements 3 and 9. It is observed from these tables that the present solution gives very sharp enclosure to the values of displacements and forces. Further, the results of displacements obtained using the present approach agree very well with the results obtained using Neumaier's approach. *It is observed that Krawczyk FPI fails to provide an enclosure at 11.5 percent uncertainty of both load and modulus of elasticity (E).* Figure 6 shows the variation of vertical displacement at node 4 with the variation of uncertainty of Young's modulus and load. Figure 7 shows the variation of axial force in element 9 with the variation of uncertainty of Young's modulus and load. *It is observed from these figures that the present solution encloses the combinatorial solution at all levels of uncertainty.*

Table 4 Eleven bar truss - comparison of axial forces for 10% uncertainty in the modulus of elasticity (E) for various approaches

	$\underline{N}_3\,(kN)$	$\overline{N}_3\,(kN)$	$\underline{N}_9\,(kN)$	$\overline{N}_9\,(kN)$
Combinatorial approach	-6.28858	-5.57152	-10.54135	-9.73966
Simple enclosure $\mathbf{z}_1(\mathbf{u})$	-7.89043	-3.96214	-11.89702	-8.39240
Error %(width)	447.83		337.15	
Intersection $\mathbf{z}_2(\mathbf{u})$	-6.82238	-5.08732	-11.32576	-9.02784
Error %(width)	141.97		186.63	
Present approach	-6.31656	-5.53601	-10.58105	-9.70837
Error %(width)	8.85		8.85	

Table 5 Eleven bar truss -displacements for 1% uncertainty in the modulus of elasticity (E) and load

	$V_2\times10^{-5}$		$U_4\times10^{-5}$		$V_4\times10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-15.09956	-14.80057	2.90870	3.01809	-0.75255	-0.72918
Neumaier's approach	-15.09972	-14.79891	2.90792	3.01836	-0.75258	-0.72898
Error %(width)	0.60		0.96		0.98	
Present approach	-15.09972	-14.79891	2.90792	3.01836	-0.75258	-0.72898
Error %(width)	0.60		0.96		0.98	

Table 6 Eleven bar truss - Axial forces for 1% uncertainty in the the modulus of elasticity (E) and load

	N_3 (kN)		N_9 (kN)	
	Lower	Upper	Lower	Upper
Combinatorial approach	-5.99198	-5.861027	-10.23567	-10.05406
Intersection $\mathbf{z}_2(\mathbf{u})$	-6.03902	-5.81438	-10.30685	-9.98368
Error %(width)	71.53		77.95	
Present approach	-5.99224	-5.86033	-10.23604	-10.05338
Error %(width)	0.72		0.58	

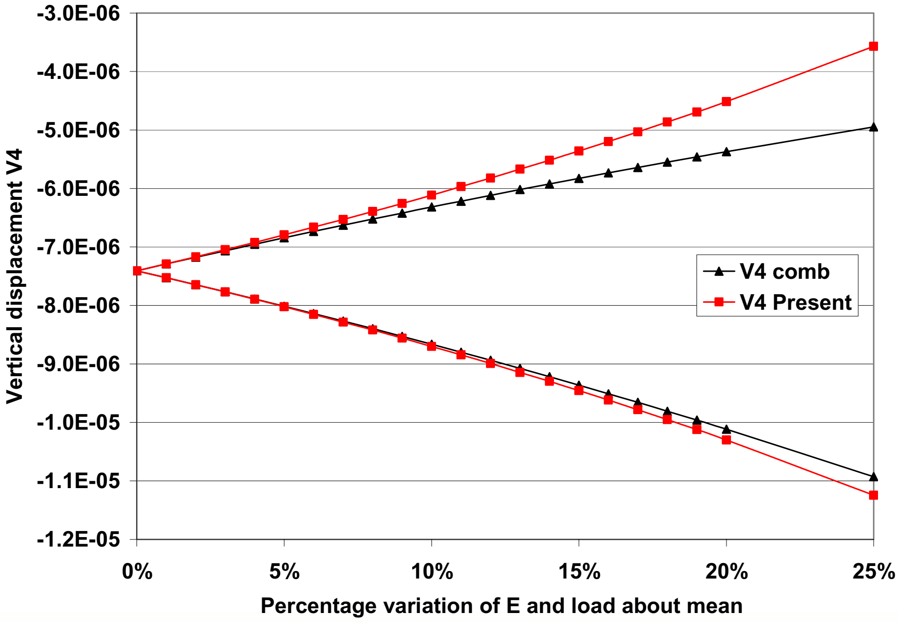

Figure 6. Eleven bar truss – Variation of vertical displacement at node 4 with uncertainty of load and E. Comb. solution vs present solution.

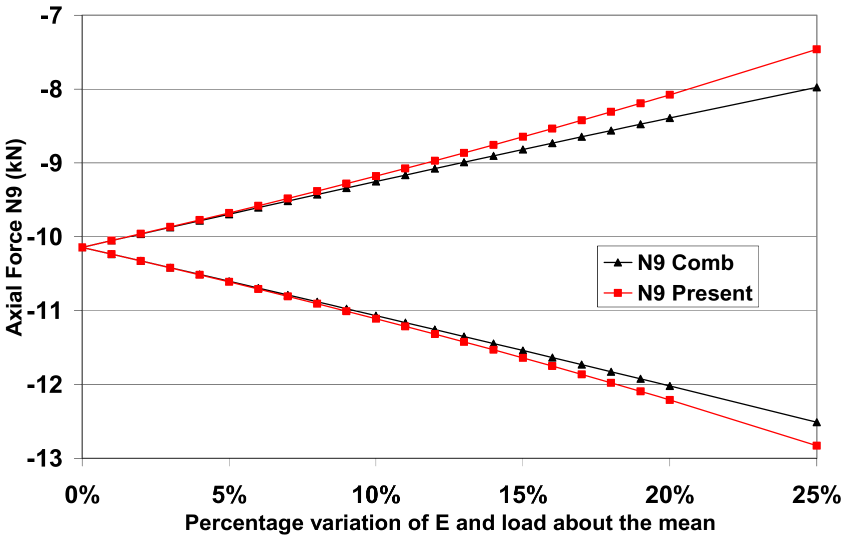

Figure 7. Eleven bar truss – Variation of axial force in element 9 with uncertainty of load and E. Comb. solution vs, present solution.

The cantilever truss structure as described earlier is shown in Figure 8. This structure has 101 elements with interval modulus of elasticity (E). The numerical values adopted to analyse the truss are P= 1000 N, L = 1 m, A = 0.01 m^2, and E = 2×10^{11}N/m^2. Maximum uncertainty allowed in modulus of Elasticity (E) is 5 percent ($\pm2.5\%$ from the mean value of E). Four different solutions are presented for the problem using the following approaches

- Element-by-element (EBE) method (Muhanna, Zhang and Mullen, 2007)
- Neumaier's method (Neumaier and Pownuk, 2007)
- Pownuk's sensitivity analysis (Pownuk, 2004) and
- Present approach.

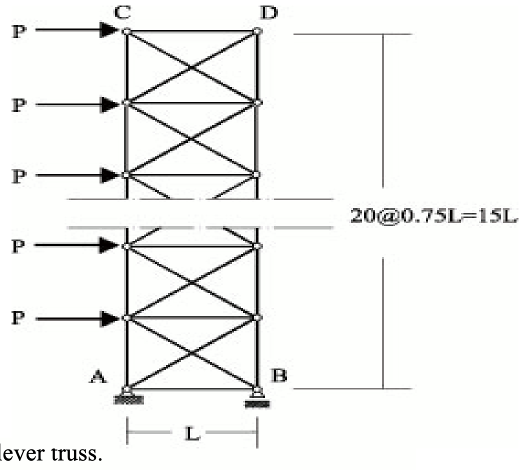

Figure 8. Benchmark problem – Cantilever truss.

The results for the horizontal and vertical displacement at the right upper corner (node D) of the truss are computed. These values are used for the computation of non-dimensional constants C_{xD} and C_{yD} defined as $C_{xD} = U_D\left(\frac{AE}{PL}\right)$ and $C_{yD} = V_D\left(\frac{AE}{PL}\right)$. Values of C_{xD} and C_{yD} are computed for all the approaches and are presented in Tables 3 and 4. It is observed from Tables 7 and 8 that widths of enclosures obtained using the present approach agree very well with the Neumaier's approach as well as sensitivity analysis. The variation of the width of the enclosure with uncertainty of Young's modulus for displacements U_D and V_D can be observed from the Figures 9 and 10. It is further observed that the EBE method gives overestimated results compared to the other three methods. Further, it is to be noted that sensitivity analysis provides an inner bound solution to the displacements.

Table. 7 Cantilever truss- non-dimensional width C_{xD} of displacement U_D

Uncertainty (%)	0.0	1.0	2.0	3.0	4.0	5.0
EBE approach	0.0	218.50	535.30	1002.7	1700.2	2747.80
Neumaier's approach	0.0	183.16	368.34	555.71	745.3	937.20
Present approach	0.0	183.12	368.33	555.67	745.19	936.95
Sensitivity analysis	0.0	182.10	364.20	546.4	728.65	911.10

Table.8 Cantilever truss - non-dimensional width C_{yD} of displacement V_D

Uncertainty (%)	0.0	1.0	2.0	3.0	4.0	5.0
EBE approach	0.0	14.48	46.03	104.84	206.84	376.89
Neumaier's approach	0.0	8.41	16.92	25.54	34.28	43.14
Present approach	0.0	8.40	16.91	25.53	34.26	43.11
Sensitivity analysis	0.0	8.35	16.70	25.06	33.41	41.78

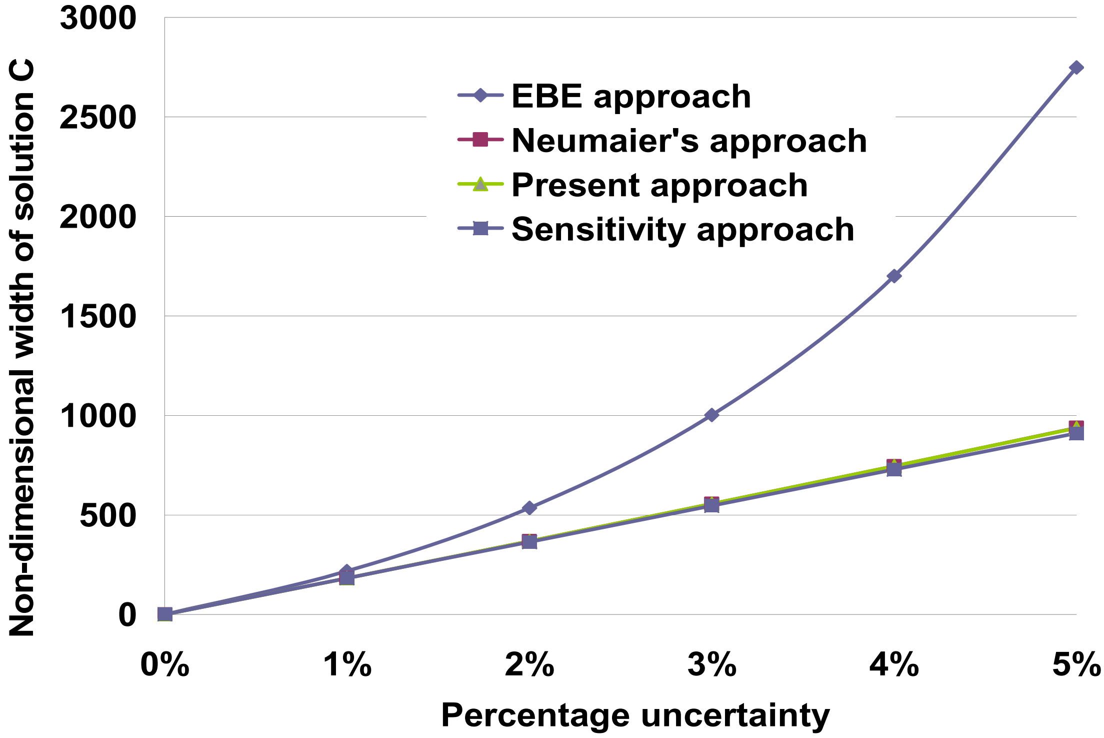

Figure 9. Cantilever truss - variation of non-dimensional widths of inte val solution of U_D w.r.t. uncertainty of E.

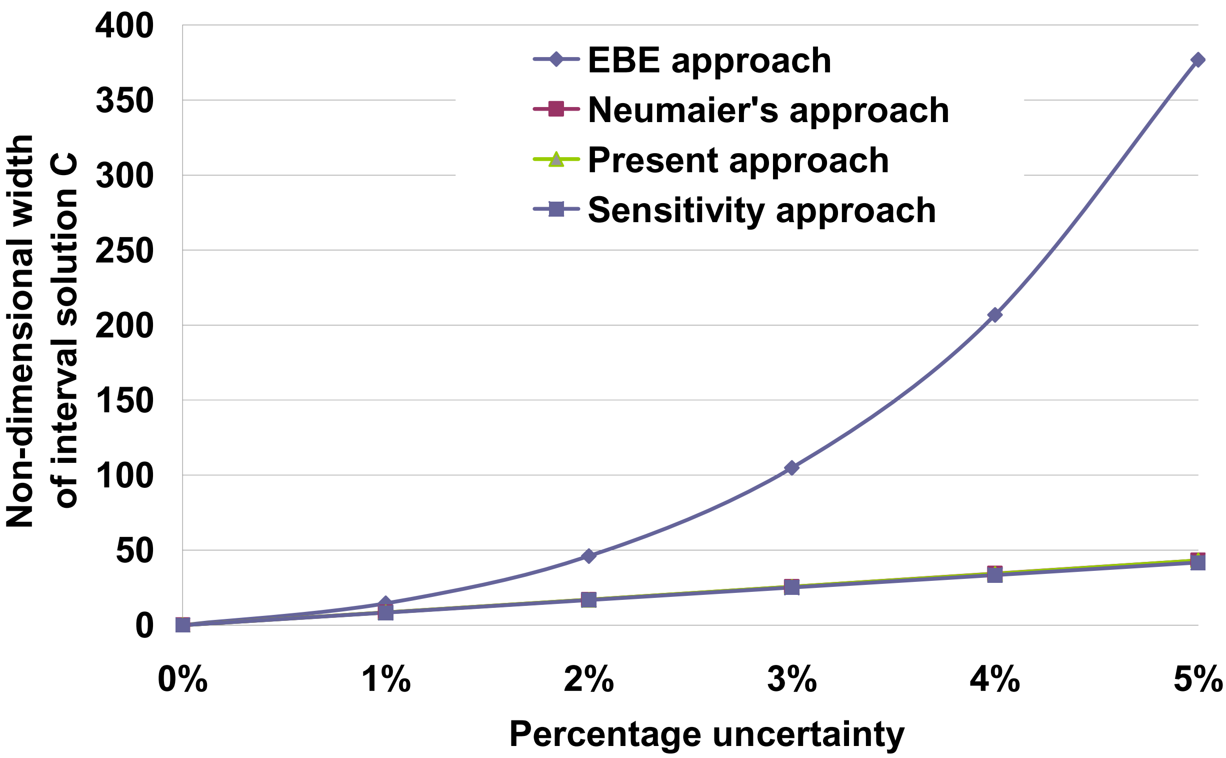

Figure 10. Variation of non-dimensional widths of interval solution of V_D w.r.t. uncertainty of E.

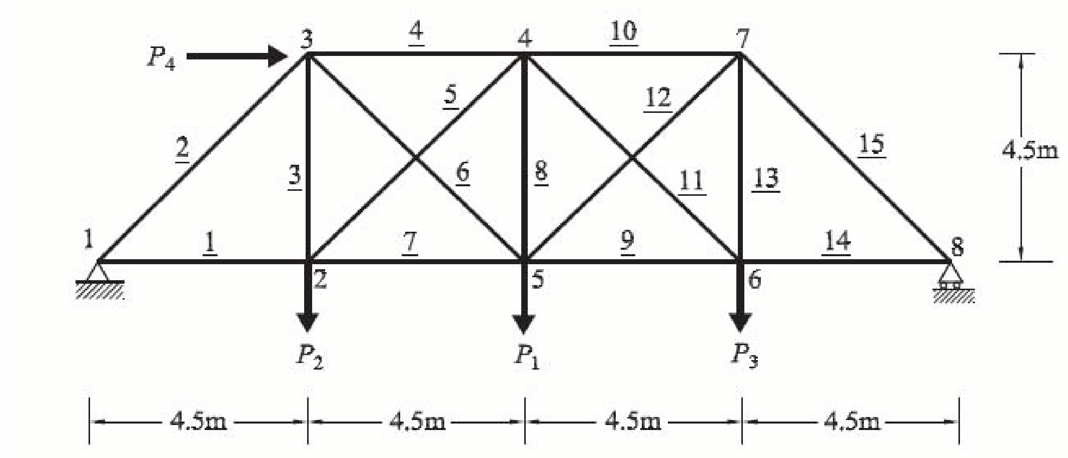

Figure 11. Fifteen bar truss.

The fifteen element truss shown in Figure 11 is subjected to vertical point loads of P_1=200 kN, P_2=100 kN,P_3=100 kN and a horizontal point load P_4=90 kN applied at the joints 5, 2, 6 and 3 respectively. Cross section areas of elements 1,2,3,13,14 and 15 are 10.0×10^{-4} m^2 while for the rest of the elements is the cross sectional area is 6.0×10^{-4} m^2. The deterministic value of Young's modulus of each element is $E_i=2\times10^{11}N/m^2$, i =1, 2 ,…15, while the cross sectional area is 0.01 m^2. The modulus of elasticity of each element is assumed to vary independently. Results are computed using combinatorial approach, Neumaier's approach and present approach. The following two case studies are taken up to demonstrate the effectiveness of the present approach.

- Case A: 10 percent uncertainty in modulus of Elasticity (E) while loads are deterministic.
- Case B: 10 percent uncertainty in both modulus of Elasticity (E) and loads

Table 9 presents the axial forces in elements 1,2,14 and 15 for case A. It is observed that the forces in these elements computed using the present approach as well as combinatorial solution are thin intervals and match exactly with each other. This is because the axial forces in these elements are independent of structural stiffness. However, it is observed from Table 9 that forces in these elements obtained using Neumaier's approach have interval values. *Thus Neumaier's approach fails to capture the deterministic nature of axial forces in these elements.* Error in bounds is computed because error in width can not be computed owing to zero width of combinatorial solution. Table 10 shows the corresponding axial forces in the elements 3, 4 and 5. It is further observed that the solution obtained by present approach matches exactly with the combinatorial solution while Neumaier's approach gives an overestimated solution.

Table 9 Forces (kN) in elements 1,2, 14 and 15 of fifteen element truss for 10% uncertainty in the modulus of Elasticity

	N_1 (kN)		N_2 (kN)		N_{14} (kN)		N_{15} (kN)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	267.500	267.500	-251.022	-251.022	222.500	222.500	-314.662	-314.662
Neumaier's approach	240.750	295.660	-277.450	-225.920	200.250	245.92	-347.780	-283.200
Error %(bounds)	-10.00	10.53	10.53	-10.00	-10.00	10.53	10.52	-10.00
Present approach	267.500	267.500	-251.022	-251.022	222.500	222.500	-314.662	-314.662
Error %(bounds)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 10 Forces (kN) in elements 3,4 and 5 of fifteen element truss for 10% uncertainty in the modulus of Elasticity (E)						
	N_3 (kN)		N_4 (kN)		N_5 (kN)	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	115.221	126.900	-329.778	-318.099	-38.043	-21.526
Neumaier's approach	103.379	139.929	-361.014	-288.885	-41.594	-18.910
Error %(width)	114.609		-330.390		-38.858	
Present approach	114.609	127.476	-330.390	-317.523	-38.858	-20.661
Error %(width)	10.17		10.17		10.17	

Table 11 depicts the horizontal and vertical displacements U_5 and V_5 of node 5 for case B. It is observed from Table 11 that, even for a large uncertainty of 10 percent, the displacements obtained using the present approach give excellent bounds to combinatorial solution and also compare very well with the results of Neumaier's approach.

Table 11 Fifteen element truss –displacements for 10% uncertainty i the modulus of elasticity (E) and load				
	$U_5 \times 10^{-2}$		$V_5 \times 10^{-2}$	
	Lower	Upper	Lower	Upper
Combinatorial approach	1.51430	1.87214	-6.56423	-5.37344
Neumaier's approach	1.49353	1.87429	-6.57150	-5.30663
Error %(width)	6.40		6.22	
Present approach	1.49353	1.87429	-6.57150	-5.30663
Error %(width)	6.40		6.22	

Table 12 presents the axial forces in selected elements for case B. It is observed from Table 12 that the widths of the axial forces in elements 1,2,14 and 15 computed using Neumaier's approach are quite large in while the corresponding errors in widths are zero in the case of forces obtained using present approach. Also, the axial forces in elements 3, 4 and 5 computed using the present approach provide a sharp enclosure to combinatorial solution while Neumaier's approach provides overestimated bounds to combinatorial solution. Thus it is concluded that the present approach provides very sharp enclosures to the axial forces obtained using combinatorial solution while Neumaier's solution provides overestimated bounds.

Table 12 Forces (kN) in elements of fifteen element truss for 10% uncertainty in modulus of elasticity (E) and load								
Element	Combinatorial approach		Neumaier's approach		%Error in width	Present approach		%Error in width
	LB	UB	LB	UB		LB	UB	
1	254.125	280.875	227.375	310.440	210.53	254.125	280.875	0.000
2	-266.756	-235.289	-294.835	-210.187	169.01	-266.756	-235.289	0.000
3	108.385	134.257	95.920	148.174	101.97	107.098	134.987	7.797
4	-346.267	-302.194	-379.167	-272.461	142.12	-347.003	-300.909	4.585
5	-43.854	-16.275	-48.143	-12.985	27.48	-44.975	-14.543	10.344
14	211.375	233.625	189.125	258.217	210.53	211.375	233.625	0.000
15	-330.395	-298.929	-365.174	-267.463	210.53	-330.395	-298.929	0.000

4. Conclusions

A new formulation for Interval Finite Element Methods is introduced. In this approach, both primary and derived quantities of interest are included in the original uncertain system as primary variables. The formulation is based on the variational approach and Lagrange multiplier method involving imposition of certain constraints that allow the Lagrange multipliers them-selves to be the derived quantities. Numerical results of this new formulation are illustrated in a number of example problems. It is observed that the displacements obtained by present approach provide a sharp enclosure to combinatorial solution and agree very well with the results obtained by Neumaier at all uncertainties. However, Krawczyk's Fixed Point Iteration fails to provide any enclosure to the solution at large uncertainties. Further, the forces computed using the present approach provide sharp enclosure to the combinatorial solution while forces computed using the approach suggested by Neumaier are found to yield significantly overestimated results. The present approach captures exactly the behaviour of statically determinate structures where the internal forces in elements are independent of material properties while all previous methods do not.

The present method addresses the basic issue of eliminating the additional overestimation in the derived quantities by adopting the mixed formulation that makes possible the simultaneous computation of primary and derived variables at the same level of accuracy. The present approach can find further application in the area of non-linear problems of structural mechanics involving large uncertainties of structural parameters.

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