Finite Element Structural Analysis using Imprecise Probabilities Based on P-Box Representation

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Abstract: Imprecise probability identifies a number of various mathematical frameworks for making decisions when precise probabilities (or PDF) are not known. Imprecise probabilities are normally associated with epistemic sources of uncertainty where the available knowledge is insufficient to construct precise probabilities. While there is no "unified theory of imprecise probabilities", most frameworks describe the uncertainty in terms of bounded possibilities or unknown probabilities between a specified lower and upper bounds.
In this paper, the authors' previous developed interval finite element methods are extended to compute p-box structures of a finite element solution where loading parameters are described by p-box structures. Both discrete p-box structures and interval p-box Monte Carlo algorithms are presented along with example problems that illustrate the capabilities of the new methods. The computational efficiency of the p-box finite element methods is also presented.

Keywords: Uncertainty; Imprecise Probability; P-Box; Interval Finite Elements.

1. Introduction

Engineering analysis and design require users to make several assumptions, typically at different levels. One of these levels is the choice of the underlying mathematical model of the system. Another level is the description of the model parameters. Assumptions are usually made to facilitate processing the analysis and design, which result in that the nature of engineering computation is conditioned by a priori assumptions. In a deterministic analysis, the geometry, loads, and material properties are assumed to have specific values. Conversely, if these assumptions do not hold, a new model with a new set of assumptions has to be used to reflect the real variations in geometry, load and material properties. The best candidate methods for handling this new situation are probabilistic methods. However, they assume complete information about the variability of all parameters and their respective Probability Density Functions (PDFs) are available. The available information, however, might range from scarce or limited to comprehensive. When there is limited or insufficient data, designers fall back to deterministic analysis, which is an irrational approach that does not account for parameters variability. On the other hand when more data is available but insufficient to completely justify a particular PDF, designers would assume a probability distribution that might not represent the real behavior of the system. More advanced Bayesian methods can treat the incomplete information in describing PDFs by updating parameters of a probability distribution. With a Bayesian approach, subjective judgments are required to estimate the random variables (i.e., form of the PDF), and thus the approach remains a subjective representation of uncertainty. In such a case, we find ourselves in

4th International Workshop on Reliable Engineering Computing (REC 2010)
Edited by Michael Beer, Rafi L. Muhanna and Robert L. Mullen

ISBN: 978-981-08-5118-7. *Published by* Research Publishing Services.
doi:10.3850/978-981-08-5118-7_013

the extreme either/or situation: deterministic or probabilistic analysis; the former does not reflect parameters variability and the latter is only valid under the assumption of the form of the PDF. The above discussion illustrates the challenge that engineering analysis and design is facing in how to circumvent the situations that do not reflect the actual state of knowledge of considered systems and based on unjustified assumptions. The current paper will focus on methods for handling uncertainties in mathematical model parameters based on available information.

Uncertainties can be classified in two general types; aleatory (stochastic or random) and epistemic (subjective) (Yager et al., 1994; Klir and Filger, 1988; Oberkampf et al., 2001). Aleatory or irreducible uncertainty is related to inherent variability and is efficiently modeled using probability theory. When data is scarce or there is lack of information, the probability theory is not useful because the needed probability distributions cannot be accurately constructed. In this case, epistemic uncertainty, which describes subjectivity, ignorance or lack of information, can be used. Epistemic uncertainty is also called reducible because it can be reduced with increased state of knowledge or collection of more data. Formal theories to handle epistemic uncertainty have been proposed including Dempster–Shafer evidence theory (Yager et al., 1994; Klir and Filger, 1988), possibility theory (Dubois and Prade, 1988), interval analysis (Moore, 1966), and imprecise probabilities (Walley, 1991).

It is possible to represent uncertainty or imprecision using imprecise probabilities (Walley, 1991; Sarin, 1978; Weichselberger, 2000) which extend the traditional probability theory by allowing for intervals or sets of probabilities. In general, imprecise probabilities present computational challenges that should be handled. By imposing some restrictions, Ferson and Donald (1998) have developed a formal Probability Bounds Analysis (PBA) that facilitates computation; Berleant and collaborators independently developed a similar approach (Bearlant, 1993; Bearlant and Goodman-Strauss, 1998). Also, related methods were developed earlier for Dempster-Shafer representations of uncertainty (Yager, 1986). PBA can represent uncertainty or imprecision, and it has been shown to be useful in engineering design (Aughenbaugh, J. M., and Paredis, C. J. J., 2006).

PBA represents uncertainty using a structure called a probability-box, or p-box which is essentially an imprecise CDF. Upper and lower CDF curves represent the bounds between which all possible probability distributions might lie. Implementation of PBA computations by discretizing the p-box and then using algorithms, called dependency bounds convolutions (DBC) have been developed by Williamson and Downs (1990) for the binary mathematical operations of addition, subtraction, multiplication, and division. The algorithms are based on interval arithmetic (Moore, 1979) and result in bounds on the true probability distribution.

In this work we will focus on a p-box structure to represent imprecise probabilistic information and will show that imprecise probability can provide a frame work for handling incomplete information in engineering analysis. Imprecise probability technique will be applied to finite element analysis. Using imprecise probability, engineering analysis can be performed considering uncertain behavior that can be quantified without requiring specific assumptions in the form of PDF of system parameters. Conversely, if enough information is available to quantify the PDF, the additional information can be accounted for. This approach will ensure that the actual state of knowledge on the system parameters is correctly reflected in the analysis and design; hence, design reliability and robustness are achieved.

2. Formulation of the problem

Within the context of structural finite element methods the final solution is introduced in a form of a system of algebraic equation of the following shape:

$$KU = P\,, \tag{1}$$

where K, U, and P are stiffness matrix, displacement vector, and load vector, respectively. In this work, we will focus on obtaining the system response due to load uncertainty introduced in a form of a p-box. The treatment of stiffness uncertainty is still under further research. Using boldface font to represent quantities of p-box structure, equation (1) takes the form:

$$K\boldsymbol{U} = \boldsymbol{P} \tag{2}$$

The solution of equation (2) takes the form:

$$\boldsymbol{U} = K^{-1}\boldsymbol{P}\,, \tag{3}$$

where K^{-1} is the inverse of the stiffness matrix and $\boldsymbol{U}$ is the system response (displacements) obtained as a p-box. In order to explain the structure of a p-box, let $F(x)$ denote the cumulative distribution function (CDF) for the random variable X. When the distribution parameters are uncertain, for every x, an interval $[[\underline{F}(x), \overline{F}(x)]$ generally can be found to bound the possible values of $F(x)$, i.e., $\underline{F}(x) \le F(x) \le \overline{F}(x)$. Such a pair of two CDFs $\underline{F}(x)$ and $\overline{F}(x)$ construct a so-called *probability box* or *probability bounds* Ferson (2003). Fig. 1 shows the probability box for a normal distribution with an interval mean of [2.0, 3.0] and a standard deviation of 0.5. In this simple example, it is easy to verify that $\underline{F}(x)$ is the CDF of the normal variable with a mean of 3 and $\overline{F}(x)$ is the one with a mean of 2. Probability box represents a general framework to represent imprecisely specified distributions. It can represent not only distributions with unknown parameters, but also distributions with unknown type or even unknown dependencies.

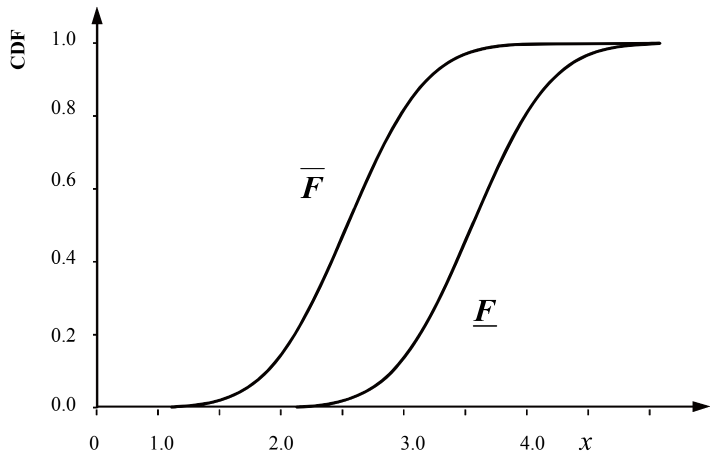

Figure 1: A probability box defined by a normal distribution with a mean of [2:0; 3:0] and a standard deviation of 0.5.

3. Interval Monte Carlo Methods

Consider the basic Monte Carlo sampling method for estimating the response statistics. Let X denotes the vector of the basic random variables representing uncertain quantities such as applied loads, material strength and stiffness in a structural mechanics problem. We are interested to estimate the statistics of some structural responses such as displacement or stress. The structural response is computed from a mathematical model, typically a finite element (FE) model for real structures, and can be conceptually expressed as $G(X)$ in which $G(\ )$ is the mapping relating the basic variables and the structural response. To estimate response statistics, sample realizations $\hat{x}_i$, $i = 1,\ldots,N$, of X are randomly generated according to the joint probability density function of X, and the corresponding structural response $G(\hat{x}_i)$ are evaluated. The response statistics (e.g., frequency distribution, sample mean, sample standard deviation) can be obtained based on the simulated samples of $G(\hat{x}_i)$.

The above Monte Carlo sampling method can be extended to p-box. When the basic variables X is modeled as p-box, the randomly sampled basic variables $\hat{x}_i$ vary in intervals. One can randomly generate such intervals using the inverse transform method (Tang 1975). Suppose a p-box which is bounded by $\overline{F}_X(x)$ and $\underline{F}_X(x)$, as shown in Fig. 2. For each realization to be generated, a standard uniform random number u_i is first generated. The intersection of a line of u_i and the lower and upper bound defining the p-box will result in an interval $[\underline{x}_i, \overline{x}_i]$. The method is graphically demonstrated in Fig. 3 for one-dimensional case. Such a pair of $\underline{x}_i$ and $\overline{x}_i$ form an interval $[\underline{x}_i, \overline{x}_i]$ which contains all possible simulated numbers from the ensemble of distributions for a particular u_i. The random interval values for the basic variables are entered into the FE model. Conceivably, the structural response $G(\hat{x}_i)$ itself varies in an interval. If the minimum and the maximum values of $G(\hat{x}_i)$ can be determined then we can plot the frequency distributions of the minimum and maximum values of the structural response, i.e., the frequency distributions for $Min(G(\hat{x}_1)), \cdots, Min(G(\hat{x}_N))$ and $Max(G(\hat{x}_1)), \cdots, Max(G(\hat{x}_N))$. The obtained two frequency distributions represent the probability bounds of the structural response. Further details on the interval Monte Carlo method are available in Zhang et al. (2010).

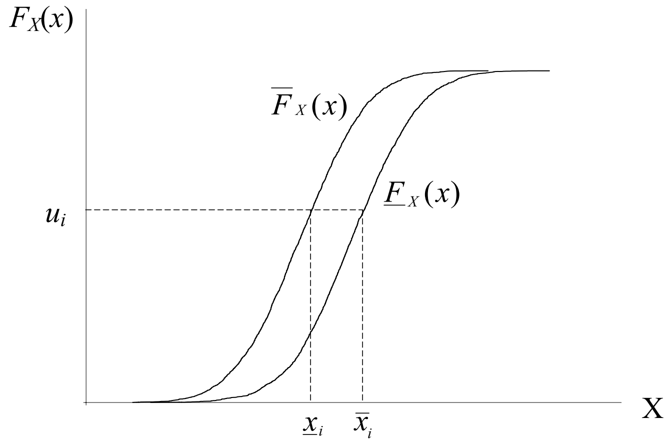

Figure 2. Generation of random intervals from a probability-box.

4. Direct P-Box calculations

An alternative to Monte Carlo p-box structural analysis is the use of discrete, interval based, p-box structures as a data type and a library of extensions of standard arithmetic operators on such structures. A discrete p-box structure consists of a collection of intervals values, each of which has an associated probability.

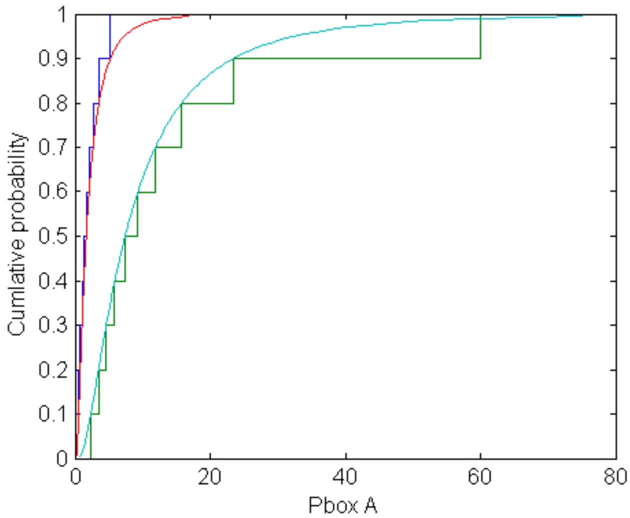

Figure 3. A p-box defined by bounding lognormal distribution with mean of [2.47, 11.08] and standard deviation of [2.76, 12.38].

In this work, we will use a uniform discretization. Thus, the associated probability for each interval is the same. For example, consider a p-box defined by bounding lognormal distribution with interval mean of [2.47, 11.08] and interval standard deviation of [2.76, 12.38]. In figure 3, the continuous p-box and the interval discretization of the p-box into 10 intervals is shown for these bounding lognormal distributions. While the continuous distributions are not bounded to the right, the discrete p-box is truncated when the cumulative distribution is greater than a specified value. In this example, we used a value of 0.99.

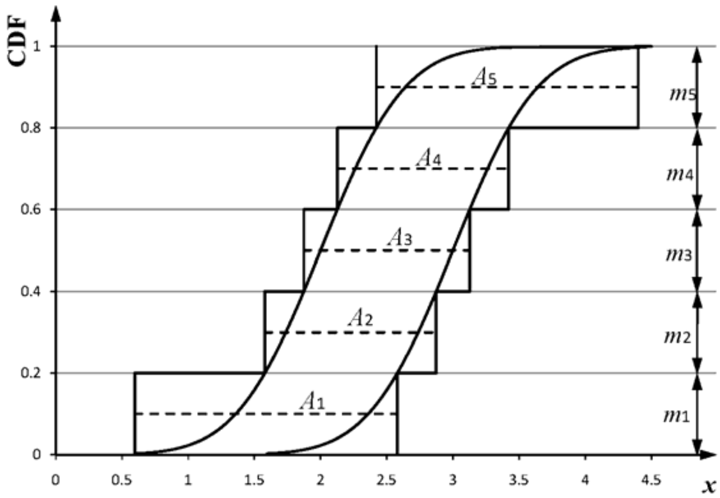

Figure 4a. Enclosure discretization.

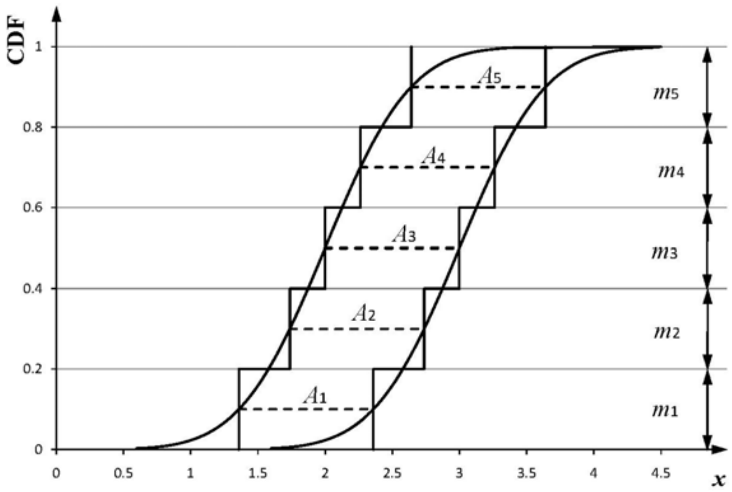

Figure 4b. Mid-point discretization.

While other researchers have used non-conservative mid-point discretization as shown in figure 4b, we have selected to use a discretization that encloses the original p-box, figure 4a.

The arithmetic of discrete p-box structures is discussed in the work of Tucker and Ferson (2003) and Williamson (1990). Williamson (1990) gives detailed description of algorithms for arithmetic operators with either the assumption of independence between variables or the consideration of any-dependency between variables. Copulas can be employed to provide for other dependency (bounds) on random variables. (Ferson 2004).

To illustrate arithmetic operations on p-boxes, we will examine the addition of two discrete p-boxes. Consider A to be the log-normal distributed p-box in Figure 3 and B to be a p-box bounded by normal distributions with means [4, 5.5] and a standard deviation of 1.5, Fig 5. The unbounded normal distributions were truncated at a probability of 0.01 and 0.99. The addition of A and B with the assumption of independence of A and B is calculated by first constructing a Cartesian product of each interval in A with B using interval arithmetic as shown in Table 1. The resulting 100 lower and upper interval values are then sorted in increasing values and corresponding pairs are used to generate 100 new interval bounds. Finally, the 100 interval discrete p-box is compressed to return the representation back to 10 interval bounds for subsequent operations. This compression is accomplished by grouping the sorted intervals into groups of 10 and replacing the 10 intervals with a single interval whose lower bound is the lower bound of the first interval in the group and whose upper bound is the upper bound of the last interval in the group. The results of this calculation are shown in Figure 6.

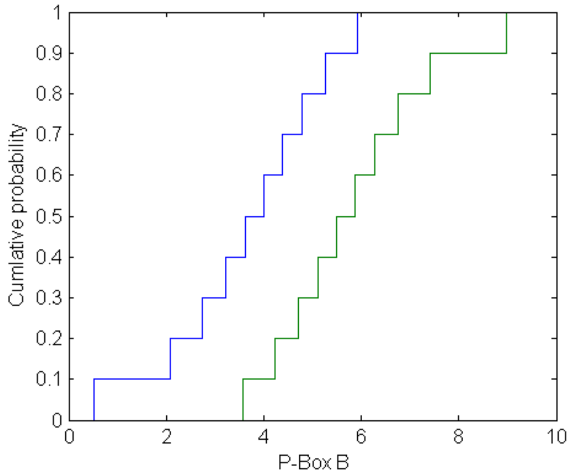

Figure 5. A p-box defined by bounding normal distribution with mean of [4.0, 4.5] and standard deviation of 1.5.

Table 1. Interval focal elements for p-boxes A and B and associated probability mass. Cartesian product for A + B

	CDF B	[0,.1]	[.1,.2]	[.2,.3]		[.7,.8]	[.8,.9]	[.9,1.0]
CDF A	A\B	[.51,3.58]	[2.08,4.24]	[2.74,4.72]	…	[4.78,6.77]	[5.26,7.43]	[5.92, 8.99]
[0.,.1]	[0, 2.33]	[.15, 5.91]	[2.08,6.57]	[2.74,7.05]	…	[4.78, 9.1]	[5.26, 9.76]	[5.92, 11.32]
[.1,.2]	[.52, 3.46]	[1.03,7.04]	[2.6,7.7],	[3.26,8.18]	…	[5.3, 10.23]	[5.78, 10.89]	[6.44, 12.45]
[.2,.3]	[.77, 4.61]	[1.28, 8.19]	[2.85,8.85]	[3.51,9.33]	…	[5.55, 1.38]	[6.03, 12.04]	[6.69, 13.6]
	⋮	⋮	⋮	⋮		⋮	⋮	⋮
[.7,.8]	[2.64,15.76]	[3.15,19.34]	[4.72,20.]	[5.3820.48]	…	[7.42, 2.53]	[7.9, 23.19]	[8.56, 24.75]
[.8,.9]	[3.52, 23.4]	[4.03, 26.98]	[5.6,27.64]	[6,26,28.12]	…	[8.3, 30.17]	[8.78, 30.83]	[9.44, 32.39]
[.9,1.0]	[5.22, 60.]	[5.73, 63.58]	[7.3,64.24	[7.96,64.72]	…	[10., 66.77]	[10.48, 67.4]	[11.44, 68.9]

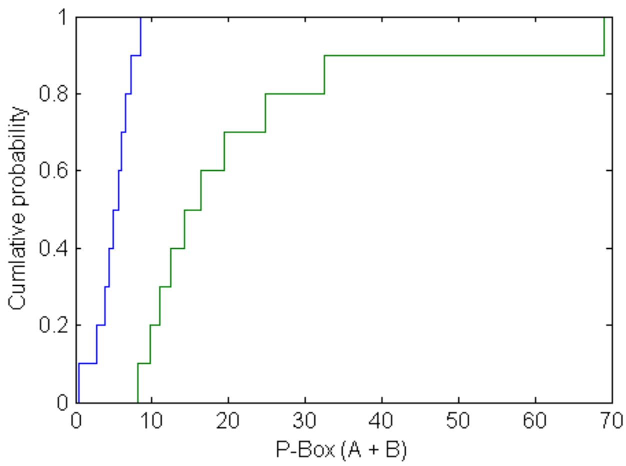

Figure 6. The sum of p-boxes A and B.

5. Results

This example is a linear elastic plane truss adopted from Halder (2000), Fig 7. The cross-sectional areas for elements A_1 to A_6 is 10.32 cm^2 and for elements A_7 to A_{15} the area is 6.45 cm^2. The elastic modulus for all the elements is 200 GPa. The statistics for the random loading is given in Table 2. Investigations of the reliability of this structure under this loading were presented by Hao et al. (2010).

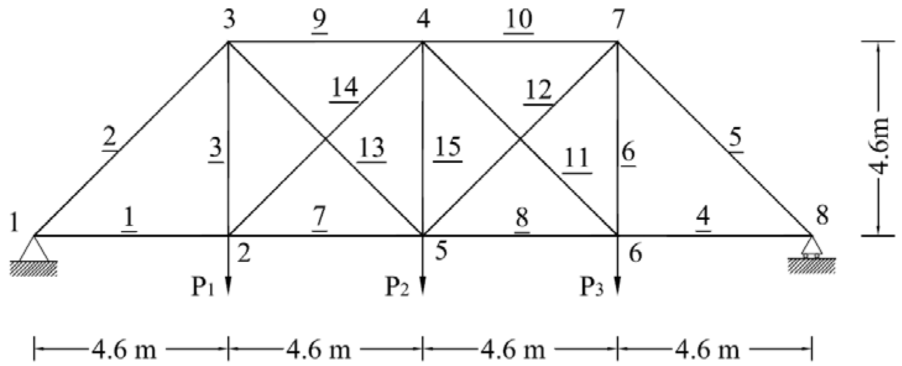

Figure 7. A truss structure

Table2. Statistics of random loadings acting on the truss		
	90% confidence interval	99% confidence interval
Mean ln P1	[4.4465, 4.5199]	[4.4258, 4.5407]
Mean ln P2	[5.5452, 5.6186]	[5.5244, 5.6393]
Mean ln P3	[4.4465, 4.5199]	[4.4258, 4.5407]
ln Standard dev. P1, P2, P3	0.09975	0.09975

5.1. CASE 1 – INTERVAL MONTE CARLO SOLUTIONS

Using the procedures given by Hao et. al. (2010), a p-box of the central displacement was constructed for 90% and 99% confidence interval bounds on the loadings.

Table 3 Central deflection of the truss lower cord -.Interval Monte Carlo results for 90% confidence interval			
Number MC simulations	Mean of the deflection of the center of the lower cord	Variance × 10^5 of the central deflection	Probability of failure
5,000	[-0.0619094, -.057528]	[1.7665,2.0459]	[0.0004,.0042]
50,000	[-0.0619236,-.0575412]	[1.7833,2.06539]	[.00024,.00446]
500,000	[-.0619299,-.057547]	[1.7724,2.053]	[.000172,.004648]

Table 3 gives the interval Monte Carlo values for the mean and variance of the deflection of the lower center node in the truss and probability of failure for different number of realizations. The exact value of the interval mean computed from moment equations is [-0.061933, -0.05755]. The Monte Carlo results for the mean are all inner bounds and the upper bound did not converge monotonically. While the estimates of the mean value were well represented by a small number of realizations, the probability of failure was not. One can observe continual monotonic improvements in the interval bounds of the probability of failure as the number of realizations were increased.

5.2. CASE 2 – DIRECT SOLUTIONS USING DISCRETE P-BOX VARIABLES

The same problem in Case 1 was analyzed using a direct calculation of the response of the structure employing a discrete p-box structure. Two different assumptions on the dependence between the loadings are used. The first is that the three applied loads are independent random values as was assumed in the Monte Carlo analysis. In addition, an assumption of any possible dependency between the three loads was also employed. Figure 8 presents the bounds on the cumulative distribution functions for the central deflection for both the 500,000 realization Monte Carlo computations as well as the direct discrete p-box computations using 200 intervals for 90% confidence interval statistics for the loadings. The Monte Carlo results are inner bounds and the discrete p-box values are guaranteed bounds. As can be seen in the figure, the distributions almost overlap except at the tails of the distributions where the Monte Carlo results are inside the discrete p-box results. This difference is, in part, due to the truncation of the p-box at the probability above 0.999.

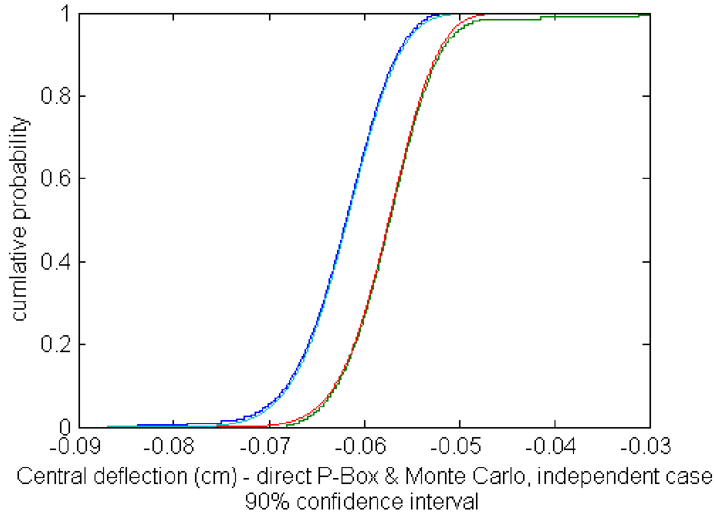

Figure 8. A p-box for central deflection of the lower cord. 90% confidence interval independent loadings is assumed. The direct computed limits are blue and green, bounds computed from Monte Carlo simulations are aqua and red.

Figure 9 presents similar results for at 99% confidence interval of the loadings. Again, the Monte Carlo and discrete p-box results match well.

Table 4 gives the computed interval means of the center deflections of the lower cord for the 90% confidence loading as a function of the number of intervals used to discretize the p-boxes. The interval means were calculated from the discrete p-box structure (Kreinovich 2007). Figure 10 shows the ratio of widths of the interval means given in table 4 to the exact mean computed by moment equations.

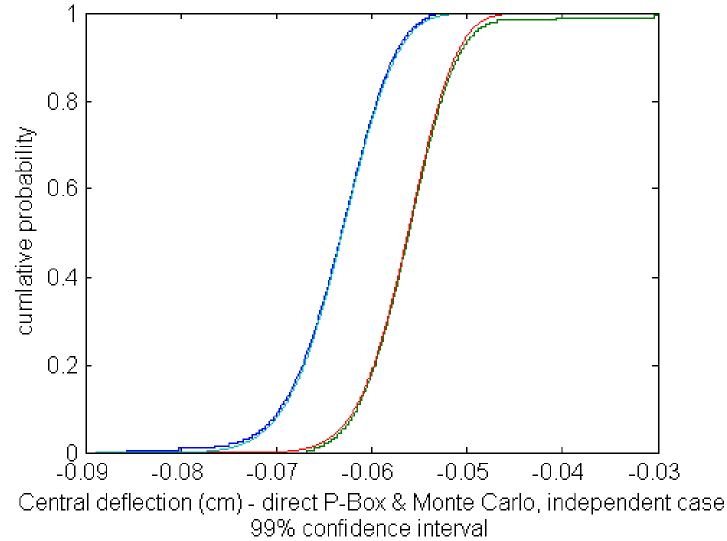

Figure 9. P-Box for central deflection of the lower cord. 99% confidence interval independent loadings is assumed. The direct computed p-box limits are blue and green; bounds computed from Monte Carlo simulations are aqua and red.

Table 4. Bounds on the mean of the central deflection for different p-box discretization levels. 90% confidence interval.

Number of p-boxs in Discretization	Mean independent	Mean any dependency	Mean Moment Equations
200	[-0.062197, -0.056952]	[-0.066677, -0.05329]	[-0.061625, -0.057264]
100	[-0.062432, -0.056385]	[-0.066699, -0.05291]	[-0.061625, -0.057264]
50	[-0.062971, -0.055293]	[-0.066993, -0.052136]	[-0.061625, -0.057264]
25	[-0.063893, -0.053213]	[-0.067388, -0.050566]	[-0.061625, -0.057264]
MC	[-0.0619123, -.0575307] (50,000 realization)		

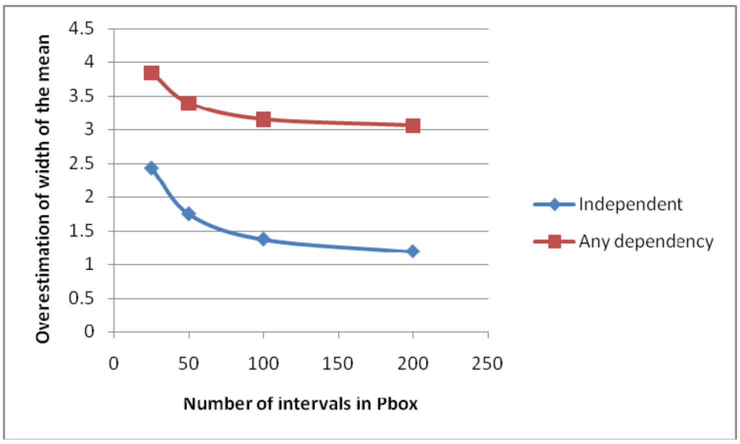

Figure 10. Ratio of interval means widths obtained from direct p-box to means obtained from moment equations for both independent and any dependency cases- 90% confidence interval.

All the computed bounds are outer bounds of the exact value of the mean (guaranteed bounds). With the assumption of independence between random variable, the width of the means computed from the p-box structure overestimate the mean values calculated from moment equations. We attribute this overestimation to the discrete p-box allowing additional CDF that were not contained in the continuous p-box. As the number of intervals used to discretize the p-box increases, the overestimation in the width of the mean approaches zero. This property is associated with the independence assumption and that in the truss problem the resulting deflections are a linear combination of the p-box structure of the prescribed loadings. The lower (upper) bounds of a linear combination of p-box structures are associated with either the upper or low bound of a loading component. The overestimation in the width of the mean in the case of any-dependency between random variables is more significant and may be more typical when the solution is not a linear combination of random variables. With the assumption of any-dependency, the p-box structure of the addition of two p-box structures is not associated exclusively with the upper or lower bounds of the operands. In the any dependency case there is an inherent overestimation of the width of the mean that is not eliminated with refinement of the discretization structure. This overestimation in the mean will still exist with sharp bounds enclosing the possible CDF. Thus this overestimation is not a direct measure of the quality of the solution.

In figures 11 and 12 we present the CDF bounds comparing the assumptions of independent loadings and any dependency loadings on the central deflection of the lower cord for the 90% and 99% confidence intervals of the loadings. The assumption of independence of the loading results in a much narrow bounds on the possible CDF.

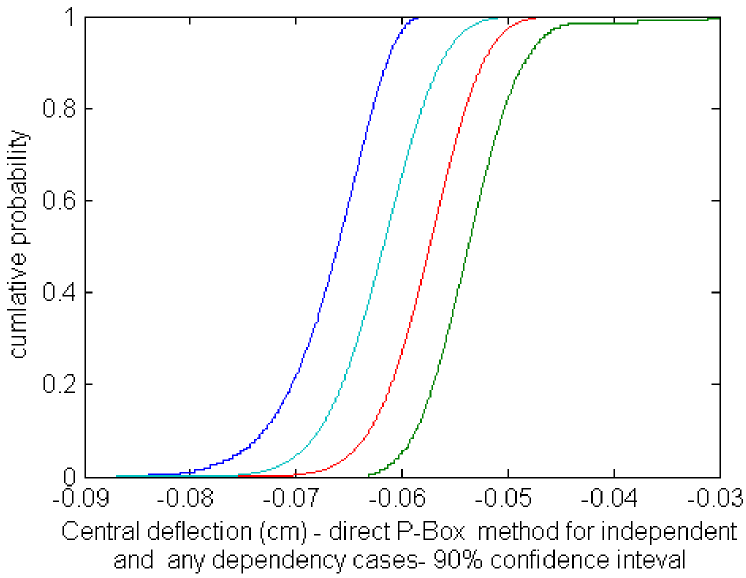

Figure 11. P-Box for central deflection of the lower cord. Any dependency between loadings variables (blue and green) compared to independent loadings (aqua and red) - 90% confidence interval.

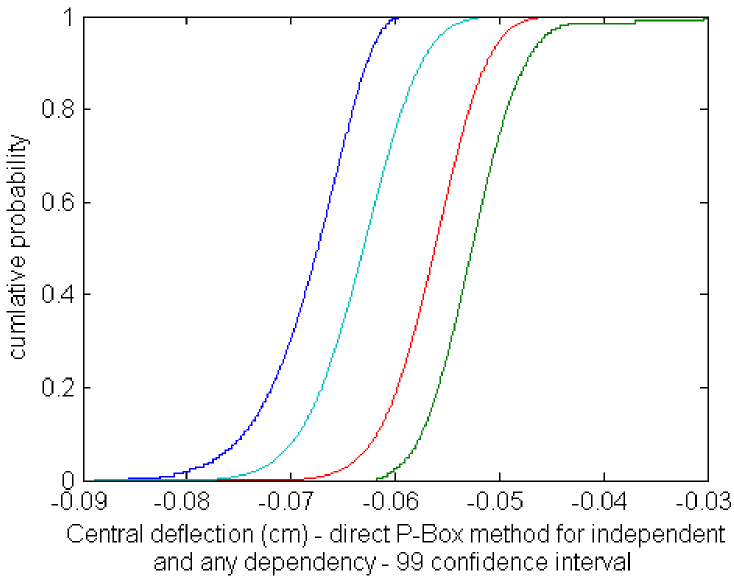

Figure 12. P-Box for central deflection of the lower cord. Any dependency between loadings variables (blue and green) compared to independent loadings (aqua and red) - 99% confidence interval.

In table 5, we compare the computational time for Monte Carlo and discrete p-box calculations. The calculations were run on an Intel® Core™2 Duo 7250 CPU @ 2.0 GHz with a single thread program. Of course the results are illustrative and no claim to an optimized implementation is made.

Table 5. Computational time for Monte Carlo and discrete p-boxes

Method	CPU time, seconds.
Monte Carlo 50,000 realizations	17
Monte Carlo 500,000	180
Discrete p-box 100 intervals independent	13
Discrete p-box 100 intervals any-dependency	1
Discrete p-box 200 intervals independent	99
Discrete p-box 200 intervals any-dependency	3

In our implementation, the Monte Carlo and the discrete p-box methods have comparable computational cost. The discrete p-box assuming any-dependency between variables results in much faster computations. Monte Carlo methods for any-dependency, to our knowledge, have not been developed yet. The straight forward inner-outer loop approach would have a significantly higher computational cost than the Monte Carlo calculations assuming independence between random variables.

6. Conclusions

A new formulation for imprecise probability in finite element analysis introduced. In this approach, uncertain loading is modeled by discrete p-box structures. The method is applied to a truss problem. A solution using discrete p-box structures is compared with an interval Monte Carlo based p-box analysis under the assumption of independent random variables. In the problem considered, the discrete p-box and Monte Carlo produced very similar results for the structural response. The computational efforts were also comparable. However, discrete p-box methods can also examine behavior with other dependencies. In particular, results are also presented for the case of any dependency between random variables. In the truss considered, the bounds on the CDF of the structural response are wider under the any-dependency assumption compared to assuming independence. The computational effort required for the any-dependency assumption is significantly less than both the Monte Carlo and discrete p-box methods when independent variables are assumed.

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