Reliability-based design by adaptive quantile estimation

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Abstract: The safety factor required to achieve certain reliability turns out to be related to the quantile of the normalized performance index of interest. Quantile functions (quantiles as functions of design parameters) are therefore essential to convert a reliability constraint into the equivalent safety-factor constraint. It is shown in this paper that the estimation of these quantile functions can be achieved by fitting the tail of the normalized performance index. In the cases where the tail varies drastically with the design parameters, an algorithm is developed to find a series of probability distributions to adaptively fit the tails. Once these probability distributions are obtained, a series of quantile functions can be found to facilitate the conversion of the reliability constraint. Three examples are investigated to verify the proposed approach. The results show that the approach can effectively convert reliability constraints into equivalent safety-factor constraints.

Key words: failure probability, reliability constraints, quantile function, subset simulation

1. Introduction

Reliability-based optimization (RBO) (Enevoldsen and Sørensen, 1994; Gasser and Schüeller, 1997; Papadrakakis and Lagaros, 2002; Royset and Der Kiureghian, 2001; Jensen, 2005) has recently become an important research area because of the need of making decisions under uncertainties in engineering applications. One of the difficulties encountered in RBO is related to the reliability constraints, to directly ensure which

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during the optimization algorithm may require numerous reliability analyses. The required computational cost can be unacceptable, rendering many realistic RBO problems computationally intractable. One possible solution is to convert these reliability constraints into non-probabilistic ones by first estimating failure probability as a function of the design parameters. This approach was taken in Gasser and Schüeller (1997) and Jensen (2005), where the logarithm of such a function is assumed to be either linear or quadratic in the design parameters. The similar approach was also taken with response surface methods or surrogate-based methods (Igusa and Wan, 2003; Eldred et al., 2002).

1.1 CONNECTION BETWEEN REQUIRED SAFETY FACTOR AND TARGET FAILURE PROBABILITY

Ching (2009) proposed a novel approach to convert reliability constraints into non-probabilistic ones by using an equivalence theorem between reliability and safety factor. He showed that a reliability constraint can be converted into a safety-factor constraint, and the required safety factor is exactly the 1-P_F^* quantile of the "normalized" performance index. What follows reviews his findings. Let $Z \in \mathbf{R}^p$ be the uncertain variables of the target system and $\theta \in \mathbf{R}^q$ be the design parameters; F denotes the failure event: F={R(Z,θ)>1}, where R(Z,θ) is called the performance index; Ř(θ) is a "nominal" performance index: an example of Ř(θ) is to take R(Z,θ) but fix Z at certain nominal values. The safety-factor approach of design is to enforce the following constraint:

$$\eta^*(\theta) \cdot \breve{R}(\theta) \le 1 \tag{1}$$

where $\eta^*(\theta)$ is the required safety factor; in general, it may depend on θ. On the other hand, the reliability-based design approach is to enforce the following constraint during the design process:

$$P\left(R[Z,\theta] > 1 \middle| \theta\right) \le P_F^* \tag{2}$$

A theorem developed in Ching (2009) states that the two constraints in (1) and (2) are equivalent if the safety factor $\eta^*(\theta)$ is found by solving the following relation:

$$P\left(G(Z,\theta) > \eta^*(\theta) \,|\, \theta\right) = P_F^* \tag{3}$$

where G(Z,θ)=R(Z,θ)/Ř(θ) is the "normalized" performance index. Note that $\eta^*(\theta)$ is simply the 1- P_F^* quantile of G(Z,θ). The proof of this theorem can be found in Ching (2009). Therefore, finding the required safety factor corresponding to a certain target failure probability is equivalent to finding a quantile of $G(Z,\theta)$.

The aforementioned theorem is practical only when $\eta^*(\theta)$ does not vary with θ, or equivalently, when the quantiles and hence the distribution of $G(Z,\theta)$ do not vary with θ; otherwise, the problem of determining a θ-dependent quantile $\eta^*(\theta)$ may be just as difficult as the original RBO problem. In the case where $\eta^*(\theta)$ is a constant η^*, the η^*-P_F^* relation can be found by the following equation:

$$P\left(G\left(Z,\theta\right) > \eta^*\right) = P_F^* \tag{4}$$

Note that θ has been removed from the condition since the conditional probability does not depend on θ. In fact, θ can be fictitiously treated as random and uniformly distributed over a prescribed allowable design region.

The key to the success of the theorem developed in Ching (2009) is a proper choice of the nominal performance function $\check{R}(\theta)$: such a proper choice can make the distribution (or quantiles) of $G(Z,\theta)=R(Z,\theta)/\check{R}(\theta)$ invariant over θ. In Ching (2009), it is argued that finding a nominal function $\check{R}(\theta)$ such that the distribution of $G(Z,\theta)$ is roughly invariant over θ is usually not a difficult task: both $\check{R}(\theta)=R[E(Z),\theta]$ or $\check{R}(\theta)=E_Z[R(Z,\theta)]$ may be acceptable. This is because the distribution of $R[Z,\theta]/R[E(Z),\theta]$ or $R[Z,\theta]/E_Z[R(Z,\theta)]$ usually does not vary drastically with θ due to the cancellation effect between $R(Z,\theta)$ and $R[E(Z),\theta]$ (or $E_Z[R(Z,\theta)]$).

1.2 Difficult cases where the quantiles are not constant

However, there are cases where the above two choices of $\check{R}(\theta)$ are not proper, i.e. the quantiles of the resulting $G(Z,\theta)$ vary significantly with θ. For these cases, the required safety factor to achieve a target failure probability P_F^* would change with the design scenario θ, rendering the theorem not practical.

A slight modification of the theorem may resolve the aforementioned issue. Suppose the distribution of $G(Z,\theta)$ varies with θ, but suppose there exists a monotonically increasing mapping L_θ parameterized by θ such that the distribution of $L_\theta[G(Z,\theta)]$ is invariant over θ, i.e. such a L_θ mapping somehow counteracts the effect of $G(Z,\theta)$. In this case, $\eta^*(\theta)$ will not be a constant but $L_\theta[\eta^*(\theta)]$ will be. This can be easily seen from the fact that

$$P_F^* = P\left(G\left(Z,\theta\right) > \eta^*\left(\theta\right) | \theta\right) = P\left(L_\theta\left[G\left(Z,\theta\right)\right] > L_\theta\left[\eta^*\left(\theta\right)\right] | \theta\right) \tag{5}$$

The last conditional probability term implies that $L_\theta[\eta^*(\theta)]$ must be a constant λ^*. The λ^* value corresponding to a target failure probability P_F^* can then be found by solving

$$P\left(L_\theta\left[G(Z,\theta)\right] > \lambda^*\right) = P_F^* \tag{6}$$

where θ has been removed from the condition and treated as random and uniformly distributed over the prescribed allowable design region. Once λ^* is found, the required safety factor is simply $L_\theta^{-1}[\lambda^*]$. Therefore, it is not necessary to solve $\eta^*(\theta)$ for each design scenario θ but only necessary to solve for the constant λ^*.

It is proposed in Ching and Hsu (2009) to take L_θ as the estimated cumulative density function (CDF) of the extreme value of G(Z,θ). This choice works because any random variable after being transformed by its CDF will be uniformly distributed over the [0,1] interval. Therefore, under this choice, the tail of $L_\theta[G(Z,\theta)]$ will be roughly uniformly distributed over [0,1] regardless the value of θ, hence the high quantiles of $L_\theta[G(Z,\theta)]$ are roughly invariant over θ.

1.3 Focus of this study

For difficult cases, it is found that a single L_θ function is usually not enough to ensure the $L_\theta[G(Z,\theta)]$ distribution to be invariant over its entire tail region. This can happen, for instance, in a problem with switching failure modes. Nonetheless, it is found that a series of $L^1_\theta, L^2_\theta, \ldots, L^m_\theta$ mappings can be more effective. This implies that solving a sequence of $\eta_1^*(\theta) < \ldots < \eta_m^*(\theta)$ that approach $\eta^*(\theta)$ may be possible, and this is the main focus of this paper.

2. Reliability Constraints

Given the design parameters θ, the probability of failure of the target system is

$$P(F \mid \theta) = \int_{\Omega_{F|\theta}} p(z \mid \theta) dz \tag{7}$$

where p(z|θ) is the probability density function (PDF) of Z; $\Omega_{F|\theta}$ is the failure domain in the $\boldsymbol{R}^p$ space: $\Omega_{F|\theta} = \{z: R(z,\theta) > 1\}$. The performance index R(Z,θ) does not necessarily define the complete collapse of the system but the performance of the system, e.g. serviceability and ultimate capacity. Throughout the paper, it is assumed without loss of generality that R(Z,θ) is positive and that p(z|θ) is known. A reliability-based optimization

(RBO) problem is to solve the following problem:

$$\min_{\theta} c_0(\theta) \qquad s.t. \qquad P(F \mid \theta) \le P_F^* \qquad c_l(\theta) \le 0 \quad l = 1, \ldots, M \tag{8}$$

where $c_0(\theta)$ is the objective function; $\{c_l(\theta)\}: l=1,..,M\}$ are deterministic constraints; $P(F|\theta)=P[G(Z,\theta)>1|\theta]\le P_F^*$ is the reliability constraint; P_F^* is the target failure probability. Note that there is another class of RBO problems where the failure probability is in the objective function. This class of RBO problems is not the focus of this paper.

The RBO problem in (8) cannot be easily solved using common optimization algorithms because of the reliability constraint. An obvious way of solving the RBO problem is to conduct a search in the θ space which may or may not require evaluating the gradients and Hessians of the functions in (8). This approach was adopted by Papadrakakis and Lagaros (2002), Tsompanakis and Papadrakakis (2004), Youn et al. (2004), etc. On the other hand, if the failure probability function P(F|θ) can be obtained beforehand as a function of θ, the reliability constraint can then be transformed into an ordinary constraint, so the RBO problem can be converted into an ordinary optimization problem that can be solved using suitable optimization algorithms. This approach was taken by Gasser and Schüeller (1997) and Jensen (2005). The method proposed in this paper should be classified as the latter.

3. Adaptive Quantile Estimation

A closer look into the theorems presented in Introduction gives deeper insights into the development of the current research. The key to the theorems lies in the transformation of the performance function R(Z,θ) so that certain distribution (or quantile) invariance occurs. For many cases, normalizing R(Z,θ) with respect to the nominal performance function $\check{R}(\theta)$ suffices, leading to (4). For more difficult cases, further application of the L_θ mapping may achieve the invariance, leading to (6). However, for difficult cases, it is usually hard to achieve the distribution invariance by applying a single L_θ mapping, and a series of $L^1_\theta, L^2_\theta, \ldots, L^m_\theta$ mappings may be needed. The algorithms of finding these $L^1_\theta, L^2_\theta, \ldots, L^m_\theta$ mappings as well as the required safety factors (i.e. the quantiles) are presented and discussed as follows.

As discussed earlier, if $P[G(Z,\theta)>\eta^*(\theta)]= P_F^*$, $\eta^*(\theta)$ is then the required safety factor corresponding to a target failure probability P_F^*, i.e. $\eta^*(\theta)$ is the $1-P_F^*$ quantile of $G(Z,\theta)$. For difficult cases, $\eta^*(\theta)$ is not a constant but varies with θ, and solving this θ-dependent quantile in a single step is extremely challenging. Nonetheless, solving a series of quantiles that approach $\eta^*(\theta)$ turns out to be simpler and is the main focus of this paper.

Let the $0.5,0.5^2,\ldots,0.5^m$ quantiles of $G(Z,\theta)$ be $\eta_1^*(\theta)$, $\eta_2^*(\theta),\ldots$, $\eta_m^*(\theta)$, i.e. $\eta_1^*(\theta)<\ldots<\eta_m^*(\theta)$ are a series of required safety factors corresponding to target failure probabilities of $0.5,0.5^2,\ldots,0.5^m$. If these quantiles $\eta_1^*(\theta)$, $\eta_2^*(\theta),\ldots$, $\eta_m^*(\theta)$ can be found and plotted against $0.5,0.5^2,\ldots,0.5^m$, the required safety-factor vs. target failure probability relation is then obtained. Given any target failure probability P_F^*, the corresponding required safety factor can then be easily identified by interpolation. Therefore, finding the required safety factor is the same as finding the quantile functions $\eta_1^*(\theta)$, $\eta_2^*(\theta),\ldots$, $\eta_m^*(\theta)$ for $G(Z,\theta)$.

Estimating quantiles for a random variable may not be new [e.g. see Deng and Pandy (2008) for a recent advancement]. However, solving the adaptive quantiles $\eta_1^*(\theta)$, $\eta_2^*(\theta),\ldots$, $\eta_m^*(\theta)$ as functions of θ can be extremely challenging. The main novelty of this research is to propose a procedure, called the adaptive quantile estimation, which takes advantage of distribution invariance to facilitate the determination of $\eta_1^*(\theta)$, $\eta_2^*(\theta),\ldots$, $\eta_m^*(\theta)$.

3.1 Estimation of the 0.5-quantile functions

Let us start the discuss of the adaptive quantile estimation based on the idea of subset simulation (Au and Beck, 2001),

$$P\left(G(Z,\theta)>\eta_2^*(\theta)\middle|\theta\right)=P\left(G(Z,\theta)>\eta_2^*(\theta)\middle|G(Z,\theta)>\eta_1^*(\theta),\theta\right)\cdot P\left(G(Z,\theta)>\eta_1^*(\theta)\middle|\theta\right) \qquad (9)$$

By definition, $\eta_1^*(\theta)$ is the 0.5-quantile of $G(Z,\theta)$, meaning that $P[G(Z,\theta)>\eta_1^*(\theta)|\theta]=0.5$ for all θ. Similarly, $\eta_2^*(\theta)$ is the $(1-0.5^2)$-quantile of $G(Z,\theta)$, meaning that $P[G(Z,\theta)>\eta_2^*(\theta)|\theta]=0.5^2$ for all θ. As a result, $\eta_2^*(\theta)$ is the 0.5-quantile of $G(Z,\theta)$ conditioning on $G(Z,\theta)> \eta_1^*(\theta)$. Continuing the same argument will lead to the conclusion that $\eta_k^*(\theta)$ is the 0.5-quantile of $G(Z,\theta)$ conditioning on $G(Z,\theta)> \eta_{k-1}^*(\theta)$.

If the CDF of $G(Z,\theta)$ conditioning on $G(Z,\theta)> \eta_{k-1}^*(\theta)$ can be found and be denoted by L^k_θ, the distribution of $L^k_\theta[G(Z,\theta)]$ conditioning on $G(Z,\theta)> \eta_{k-1}^*(\theta)$ will be invariant over θ and uniformly distributed over [0,1]. Furthermore, due to the fact that

$$0.5 = P\left(G(Z,\theta) > \eta_k^*(\theta) \middle| G(Z,\theta) > \eta_{k-1}^*(\theta),\theta\right) = P\left(L_\theta^k\left[G(Z,\theta)\right] > L_\theta^k\left[\eta_k^*(\theta)\right] \middle| G(Z,\theta) > \eta_{k-1}^*(\theta),\theta\right) \quad (10)$$

$L^k_\theta[\eta_k^*(\theta)]$ must be a constant: this is because the distribution of $L^k_\theta[G(Z,\theta)]$ does not depend on θ, so that its 0.5 quantile (namely $L^k_\theta[\eta_k^*(\theta)]$) should not depend on θ, either. In fact, $L^k_\theta[\eta_k^*(\theta)]$ is exactly 0.5 because $L^k_\theta[G(Z,\theta)]$ is uniformly distributed over [0,1]. As a consequence, the $\eta_k^*(\theta)$ function is simply $(L^k_\theta)^{-1}(0.5)$.

This is an interesting observation since it says that if the conditional CDF L^k_θ can be determined, the quantile function $\eta_k^*(\theta)$ (or the required safety factor) can be also determined. Moreover, the determination of L^k_θ requires the knowledge of the previous quantile function $\eta_{k-1}^*(\theta)$. As a result, if the $\eta_1^*(\theta)$ function is known, the quantile functions $\eta_2^*(\theta)\ldots\eta_m^*(\theta)$ can then be obtained sequentially.

3.2 Determining the CDF L^k_θ

Before presenting the determination of the CDF L^k_θ, let us explore more the distribution invariance for the problem at hand. Recall that $\eta_k^*(\theta)$ satisfies

$$P\left(G(Z,\theta) > \eta_k^*(\theta) \middle| G(Z,\theta) > \eta_{k-1}^*(\theta),\theta\right) = 0.5 \quad (11)$$

Notice that the distribution invariance already occurs in (11) since the conditional probability is a constant of 0.5 and does not depend on θ. Therefore, the θ condition in the equations can be dropped (i.e. θ is fictitiously treated as "random"):

$$P\left(G(Z,\Theta) > \eta_k^*(\Theta) \middle| G(Z,\Theta) > \eta_{k-1}^*(\Theta)\right) = 0.5 \quad (12)$$

where Θ denotes the random version of θ. In other words, $\eta_k^*(\Theta)$ is the 0.5-quantile of G(Z,Θ) conditioning on G(Z,Θ)> $\eta_{k-1}^*(\Theta)$. Once the CDF L^k_Θ of G(Z,Θ) conditioning on G(Z,Θ)> $\eta_{k-1}^*(\Theta)$ is determined, $\eta_k^*(\theta)$ can be found as $(L^k_\theta)^{-1}(0.5)$ In the following presentation, the symbol Θ will be used to denote the random version of θ, while the symbol θ will be reserved for a fixed design parameter.

The generalized Pareto distribution (GPD) is proposed to parameterize the CDF L^k_θ. This choice is justified by the fact that the tails of many distributions can be well approximated by GPD (Pickands, 1975). In more details, the CDF of the k-th stage GPD is parameterized as follows:

$$L_{\Theta}^{k}\left(g\right)=P\left(G\left(Z,\Theta\right)\le g\mid G\left(Z,\Theta\right)\ge \eta_{k-1}^{*}\left(\Theta\right)\right)=1-\left(1+\frac{\xi_k\left(g-\eta_{k-1}^{*}\left(\Theta\right)\right)}{\sigma_k\left(\Theta\right)}\right)^{-1/\xi_k} \quad (13)$$

where ξ_k is the shape parameter; $\sigma_k(\theta)>0$ is the scale parameter, made to depend on θ to model the varying-over-θ behavior, and it is taken to be linear functions of θ for simplicity, i.e. $\sigma_k(\theta)=a^0{}_k+a^1{}_k\theta_1+...+a^q{}_k\theta_q$ (q is the dimension of θ), where $a^n{}_k$ denotes the n-th component of the a_k vector. Its PDF is

$$\frac{1}{a_k^0+a_k^1\Theta_1+...+a_k^q\Theta_q}\left(1+\frac{\xi_k\left(g-\eta_{k-1}^{*}\left(\Theta\right)\right)}{a_k^0+a_k^1\Theta_1+...+a_k^q\Theta_q}\right)^{-1-1/\xi_k} \quad (14)$$

The following steps can be taken to find the CDF $L^k{}_\theta$ as well as the $\eta_k{}^*(\theta)$ function. These steps take full advantage of the distribution invariance.

(1) Obtain conditional samples of G(Z,Θ), denoted by $G_k{}^{(i)}=G(Z_1{}^{(i)},\Theta_1{}^{(i)})$, where $\{Z_1{}^{(i)},\Theta_1{}^{(i)}\}$ is the *i*-th sample drawn from the PDF $p[z,\theta|G(Z,\Theta)>\eta_{k-1}{}^*(\Theta)]$ (it is assumed that these samples can be readily sampled; the sampling procedure will be elaborated in a later section). Draw *N* samples to obtain $\{G_k{}^{(i)}:i=1,\ldots,N\}$.

(2) Find the maximum likelihood estimate for $\sigma_k(\theta)$ and ξ_k by maximizing the log-likelihood function:

$$\sum_{i=1}^{N}\left(-\log\left[a_k^0+...+a_k^q\Theta_q^{(i)}\right]-\left[1-1/\xi_k\right]\log\left(1+\frac{\xi_k\left(G_k^{(i)}-\eta_{k-1}^{*}\left(\Theta^{(i)}\right)\right)}{a_k^0+...+a_k^q\Theta_q^{(i)}}\right)\right) \quad (15)$$

(3) Denote the maximum likelihood estimate by $\left\{a_k^*,\xi_k^*\right\}$. Then the estimated $\eta_k{}^*(\theta)$ is simply

$$\eta_k^*\left(\theta\right)=L_\theta^{k^{-1}}\left(0.5\right)=\eta_{k-1}^*\left(\theta\right)+\frac{a_k^{0*}+a_k^{1*}\theta_1+...+a_k^{q*}\theta_q}{\xi_k^*}\left(2^{\xi_k^*}-1\right) \quad (16)$$

The aforementioned steps require the initiation of the $\eta_1{}^*(\theta)$ function. The $\eta_1{}^*(\theta)$ function is taken to be a constant function whose value equals to the 0.5-quantile (median) of the unconditional samples of G(Z,Θ), where {Z,Θ} are distributed as $p(z,\theta)=p(z|\theta)p(\theta)$, and $p(\theta)$ is uniform over the prescribed allowable design region. The aforementioned steps also require the ability to draw samples from $p[z,\theta|G(Z,\Theta)>\eta_{k-1}{}^*(\Theta)]$, which is described in detail in the next section.

3.3 DRAWING SAMPLES FROM $p[z,\theta|G(Z,\Theta)> \eta_{k-1}^*(\Theta)]$

Similar to subset simulation, samples can be drawn from $p[z,\theta|G(Z,\Theta)> \eta_{k-1}^*(\Theta)]$ by Markov chain Monte Carlo (MCMC), explained as follows starting from stage k = 2, 3, … At the first stage, N samples of {Z,Θ}, denoted by $\{Z_1^{(i)},\Theta_1^{(i)}:i=1,\ldots,N\}$, are drawn from p(z,θ). The $\eta_1^*(\theta)$ function is then taken to be a constant equal to the 0.5-quantile (median) of the samples $\{G_1^{(i)}:i=1,\ldots,N\}$.

The $\{Z_1^{(i)},\Theta_1^{(i)}\}$ samples satisfying $\{G(Z_1^{(i)},\Theta_1^{(i)})> \eta_1^*(\Theta)\}$ are actually distributed as $p[z,\theta|G(Z,\Theta)> \eta_1^*(\Theta)]$ ($\eta_1^*(\Theta)$ here is just a constant): there are roughly 0.5N such samples. These samples will be the mothers of the next stage. Each mother generates one offspring by MCMC: simply conduct the Metropolis-Hastings (M-H) algorithm with the stationary PDF being p(z,θ) but enforce the acceptance criterion $G(Z,\Theta)> \eta_1^*(\Theta)$. Together with the original 0.5N mothers, there are, again, N samples $\{Z_2^{(i)},\Theta_2^{(i)}:i=1,\ldots,N\}$ distributed as $p[z,\theta|G(Z,\Theta)> \eta_1^*(\Theta)]$. With these N samples and the GPD maximum likelihood method presented in the previous section, one can estimate the $\eta_2^*(\theta)$ function. Continuing doing so until the ***m***-th stage will yield the estimates of the $\eta_1^*(\theta)$, $\eta_2^*(\theta)\ldots$, $\eta_m^*(\theta)$ functions.

3.4 SUMMARY OF THE PROCEDURE

The following algorithm summarizes the procedure of the proposed approach.

1. In the first stage, obtain Monte Carlo simulation (MCS) samples $\{Z_1^{(i)},\Theta_1^{(i)}:i=1,\ldots,N\}$ from p(z,θ) and compute the corresponding $\{G_1^{(i)}=G(Z_1^{(i)},\Theta_1^{(i)}):i=1,\ldots,N\}$ samples. The $\eta_1^*(\theta)$ function is then taken to be a constant equal to the median of $\{G_1^{(i)}:i=1,\ldots,N\}$. The samples whose *G* values exceed $\eta_1^*(\theta)$ will be the mothers for the next stage.
2. Based on the 0.5N mothers, generate 0.5N offsprings by the M-H algorithm to restore N samples $\{Z_2^{(i)},\Theta_2^{(i)}:i=1,\ldots,N\}$ distributed as $p[z,\theta|G(Z,\Theta)> \eta_1^*(\Theta)]$, and compute the corresponding $\{G_2^{(i)}:i=1,\ldots,N\}$ values.
3. Find the $\{a^*_2,\xi^*_2\}$ that maximizes the following log-likelihood:

$$\sum_{i=1}^{N}\left(-\log\left[a_2^0+\ldots+a_2^q\Theta_{2,q}^{(i)}\right]-\left[1-1/\xi_2^*\right]\log\left(1+\frac{\xi_2^*\left(G_2^{(i)}-\eta_1^*\left(\Theta_2^{(i)}\right)\right)}{a_2^0+\ldots+a_2^q\Theta_{2,q}^{(i)}}\right)\right) \tag{17}$$

and simply take

$$\eta_2^*(\theta) = \eta_1^*(\theta) + \frac{a_2^{0*} + a_2^{1*}\theta_1 + ... + a_2^{q*}\theta_q}{\xi_2^*}\left(2^{\xi_2^*} - 1\right) \quad (18)$$

The samples whose *G* values exceed $\eta_2^*(\Theta)$ will be the mothers for the next stage (there are 0.5N of them).

4. Cycle Steps 2-3 for stages 2, 3, …, *m* to obtain the quantile functions $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, …$\eta_m^*(\theta)$.
5. For a specific design scenario θ, plot the $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, …$\eta_m^*(\theta)$ vs. $0.5, 0.5^2, …, 0.5^m$ relation. This is exactly the required safety factor vs. target failure probability relation. Suppose the prescribed target failure probability is P_F^*, the required safety factor can then be interpolated from the $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, …$\eta_m^*(\theta)$ vs. $0.5, 0.5^2, …, 0.5^m$ relation (preferably take logarithm for $0.5, 0.5^2, …, 0.5^m$ and P_F^* during the interpolation), and denote the interpolated safety factor by $\eta^*(\theta)$. Then the reliability constraint $P[R(Z,\theta)>1|\theta] \le P_F^*$ can be then converted into the safety-factor constraint $\eta^*(\theta)\check{R}(\theta) \le 1$.

4. Numerical Example

Consider a strip shallow foundation underlain by two soil layers: a sandy soil layer near the ground surface and a clayey soil layer underneath (see Figure 1). The thickness *H* of the clay layer is uncertain. Besides *H*, the uncertainties include the saturated unit weight γ_{sat}^{clay}, compression index C_c, re-compression index C_r, initial void ratio e_0, and over-consolidation ratio OCR of the clay and the saturated unit weight of the sand γ_{sat}^{sand} and the unsaturated unit weight of the sand γ^{sand}. The uncertain variables are modeled as follows: *H* is Gaussian with mean value = 5 m and standard deviation = 0.5 m; [γ_{sat}^{clay}, γ_{sat}^{sand}, γ^{sand}] are Gaussian random variables with means equal to [18, 20, 18] kN/m^3 and standard deviations equal to [1 1.5 1] kN/m^3; [C_c, e_0] are log-normal random variables with means equal to [0.4, 0.8] and coefficients of variation (c.o.v.) both equal to [0.2, 0.1]; C_r is equal to C_c multiplied by a coefficient uniformly distributed over the interval [0.1, 0.3]; *OCR* is uniformly distributed over the interval [1.0, 1.5]. The design parameters include the bearing pressure of the shallow foundation q (θ_1) and the width of the foundation B (θ_2) of the wall. The allowable design region is chosen to be the rectangle formed by the following two constraints: $\theta_1 \in [50\ 150]$kN/m^3 and $\theta_2 \in [1\ 4]$m.

The long-term consolidation settlement of the foundation can be calculated as follows (Das, 1990):

$$S = \begin{cases} \dfrac{C_r H}{1+e_0}\log_{10}\left(\dfrac{OCR\cdot\sigma_0}{\sigma_0}\right) + \dfrac{C_c H}{1+e_0}\log_{10}\left(\dfrac{\sigma_0+\Delta\sigma}{OCR\cdot\sigma_0}\right) & if\ \sigma_0+\Delta\sigma \geq OCR\cdot\sigma_0 \\ \dfrac{C_r H}{1+e_0}\log_{10}\left(\dfrac{\sigma_0+\Delta\sigma}{\sigma_0}\right) & if\ \sigma_0+\Delta\sigma < OCR\cdot\sigma_0 \end{cases} \tag{19}$$

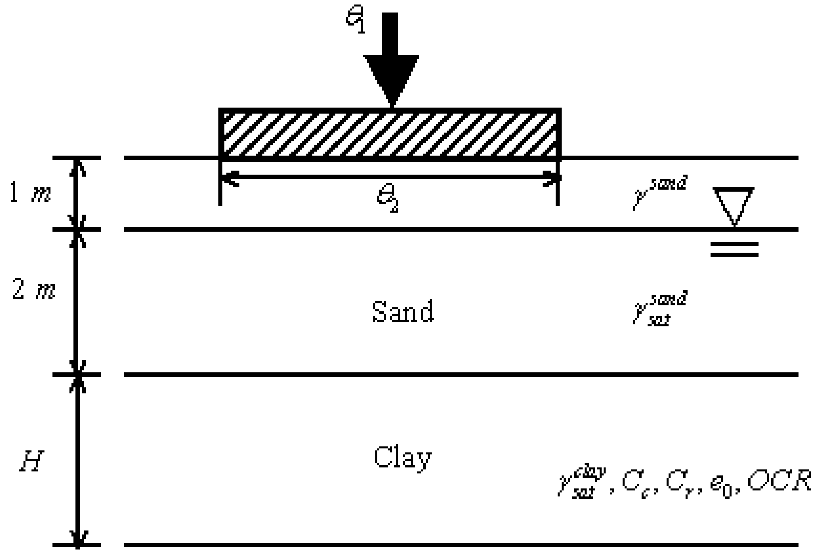

Figure 1 The cross section of the consolidation example. θ_1 and θ_2 are the bearing pressure and foundation width, respectively.

where $\sigma_0=\gamma^{sand}+2(\gamma_{sat}^{sand}-9.81)+(H/2)(\gamma_{sat}^{clay}-9.81)$ is the vertical effective stress at the middle of the clayey layer before the foundation is constructed; $\Delta\sigma$ is the vertical stress increment due to the construction of the foundation. According to the 2:1 method (Das, 1990),

$$\Delta\sigma \approx \frac{q\cdot B}{3+B+H/2} \tag{20}$$

The failure is defined as the exceedance of the consolidation settlement over 0.3 m:

$$R(Z,\theta) = S/0.3 \tag{21}$$

For this example, the nominal performance index is taken to be of the following form:

$$\breve{R}(\theta) = \exp\left[c_0 + c_1\theta_1 + c_2\theta_2 + c_{11}\theta_1^2 + c_{22}\theta_2^2 + c_{12}\theta_1\theta_2\right] \tag{22}$$

where the coefficients $c_0,c_1,c_2,c_{11},c_{22},c_{12}$ are estimated according to the following procedure: given the N MCS samples $\{\Theta_1^{(i)}:i=1,\ldots,N\}$ and $\{R_1^{(i)}:i=1,\ldots,N\}$ in the first stage of the procedure, use least-square method to fit a second-order polynomial to the sample pairs $\{(\Theta_1^{(i)},\log[R_1^{(i)}]):i=1,\ldots,N\}$ to estimate $c_0,c_1,c_2,c_{11},c_{22},c_{12}$. One can verify that the resulting $\breve{R}(\theta)$ is similar to $E_Z[R(Z,\theta)]$.

As a demonstration, the target failure probabilities of interest include 0.01, 0.001 and 0.0001. The total number m of stages is taken to be 14 since it is the smallest integer satisfying $0.5^m < 0.0001$. With a sample size of $N = 2000$, the evolutions of the estimated quantile functions $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, $\ldots\eta_m^*(\theta)$ are shown in Figure 2. Those are the required safety factors corresponding to target failure probabilities $0.5,0.5^2,\ldots,0.5^m$. Also shown in the same figure are the conditional samples $\{G(Z_k^{(i)},\Theta_k^{(i)}):i=1,\ldots,N\}$ of each stage k. The label of the vertical axis of Figure 2 is $\eta_k^*(\theta)$ for the quantile functions $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, $\ldots\eta_m^*(\theta)$ but is $G(Z,\theta)$ for the $G(Z_k^{(i)},\Theta_k^{(i)})$ samples. Judging from the spread of the samples, it is clear that the distribution of $G(Z,\theta)$ varies significantly with θ: near the corner of θ_1 = 50kN/m^2 and θ_2 = 1m, the distribution of $G(Z,\theta)$ seems to have a thicker tail for large G values, but the opposite (thinner tail) is true near the corner of θ_1 = 150kN/m^2 and θ_2 = 4m.

This phenomenon is explained in Figure 3: for the scenario of θ_1 = 150kN/m^2 and θ_2 = 4m, the clay under the foundation is mostly normally consolidated due to the large bearing pressure; the settlement is mostly due to consolidation along the C_c line in Figure 3. Therefore, there is only a dominant mode of consolidation, as seen from the single peak in the histogram of $G(Z, \theta_1=150,\theta_2=4)$. However, for θ_1 = 50kN/m^2 and θ_2 = 1m, the clay can be either over or normally consolidated since the bearing pressure is not large enough; the settlement can be due to consolidation along either the C_c or C_r line. Therefore, there are two modes of consolidation: as seen in the histogram of $G(Z, \theta_1=50,\theta_2=1)$, there seem to be two local peaks. Moreover, the right tail of $G(Z, \theta_1=50,\theta_2=1)$ is quite thick because the second peak hides in this region. This is why in Figure 2 near the corner of θ_1 = 50kN/m^2 and θ_2 = 1m, the distribution of $G(Z,\theta)$ has a thicker tail. The consequence is that for the same level of target failure probability, the required safety factor for the scenario of θ_1 = 50kN/m^2 and θ_2 = 1m is much larger than for the scenario of θ_1 = 150kN/m^2 and θ_2 = 4m.

Based on the estimated quantile functions $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, …$\eta_m^*(\theta)$, the required safety factors for $P_F^* = 0.01$, 0.001, 0.0001 can be readily interpolated. Taking 0.01 as an example, since 0.01 is between 0.5^6 and 0.5^7, the required safety factor for target failure probability of 0.01 can be interpolated as

$$\eta^*(\theta) = \frac{\left[\log(0.01) - \log(0.5^7)\right]\eta_6^*(\theta) + \left[\log(0.5^6) - \log(0.01)\right]\eta_7^*(\theta)}{\log(0.5^6) - \log(0.5^7)} \tag{23}$$

and similar for the required safety factors of target failure probabilities 0.001 and 0.0001. The required safety factors for $P_F^* = 0.01$, 0.001, 0.0001 are shown in Figure 4.

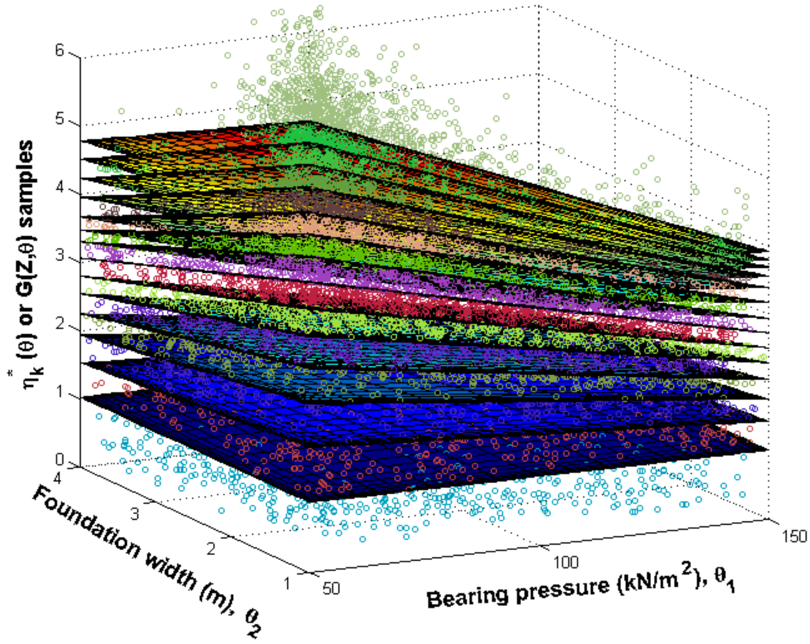

Figure 2 The evolutions of the estimated quantile functions $\eta_1^*(\theta)$, $\eta_2^*(\theta)$, …$\eta_m^*(\theta)$ for the consolidation example and the conditional samples of all stages.

According to the theorem, the reliability constraint P(F|θ)≤0.01 can be converted into the following safety-factor constraint:

$$\frac{\left[\begin{array}{l}\left[\log\left(0.01\right)-\log\left(0.5^{7}\right)\right]\eta_6^*\left(\theta\right) \\ \qquad +\left[\log\left(0.5^{6}\right)-\log\left(0.01\right)\right]\eta_7^*\left(\theta\right)\end{array}\right]}{\log\left(0.5^{6}\right)-\log\left(0.5^{7}\right)}\cdot\exp\left[c_0+c_1\theta_1+c_2\theta_2+c_{11}\theta_1^2+c_{22}\theta_2^2+c_{12}\theta_1\theta_2\right]\le 1 \qquad (24)$$

and similar for the cases of target failure probabilities 0.001 and 0.0001.

The following brute-force approach is adopted to verify the results from the proposed approach. The allowable design region is filled with dense grid points, and at each grid point, the failure probability is estimated with large-sample MCS, then all the grid points whose failure probability estimates are less than P_F^* are found. The set containing all these grid points is the actual feasible set satisfying the reliability constraint, named the reliability feasible set and denoted by Σ_R. This set will be compared with the set satisfying $\eta^*(\theta)$ $\check{R}(\theta)\le 1$, i.e.: the approximation made by the proposed approach, named the safety-factor feasible set and denoted by Σ_S. If the safety-factor feasible set of a performance index is close to the reliability feasible set, the new approach is then verified to be effective for that performance index.

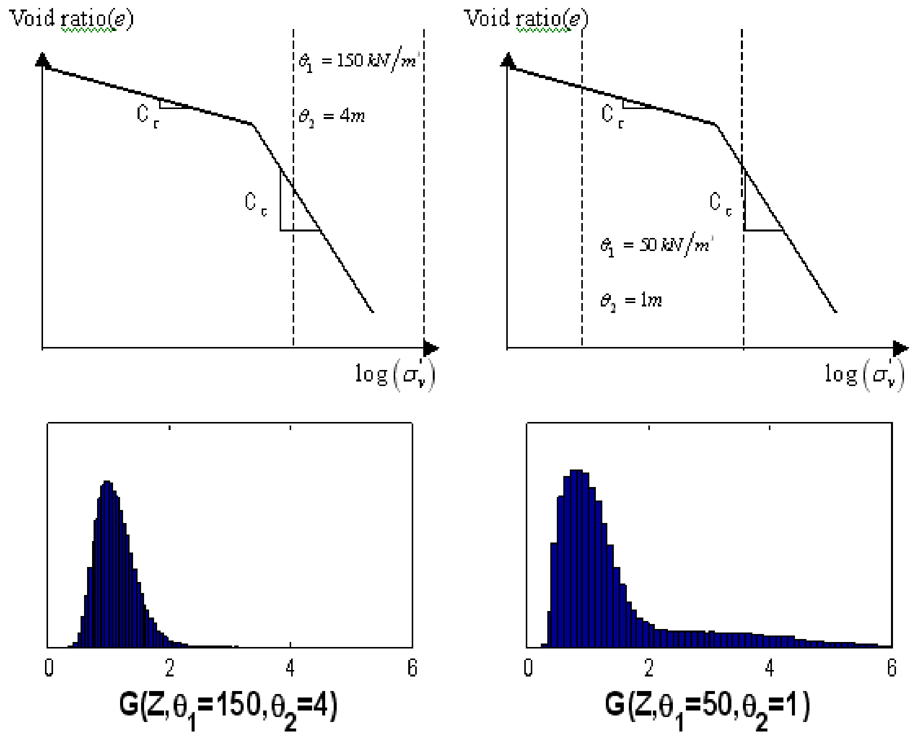

Figure 3 The scenarios of θ_1 = 150kN/m^2 and θ_2 = 4m (left column) and θ_1 = 50kN/m^2 and θ_2 = 1m (right column): the first row plots the consolidation curves, while the second row plots the histograms of the G(Z,θ) samples.

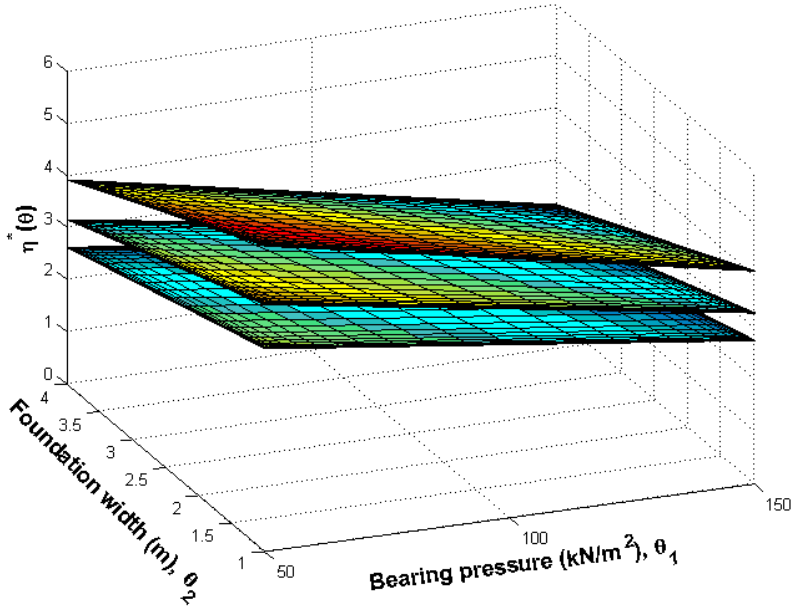

Figure 4 The required safety factors for $P_F^* = 0.01, 0.001, 0.0001$ (from lower to upper planes) for the consolidation example.

In Figure 5, the safety-factor feasible sets [sets satisfying (24)] are shown as shaded regions for various target failure probabilities. The results obtained from the old theorem developed in Ching (2009), i.e. (4), are also shown in the figure for comparison. The borders of the feasible sets include several break points: this is due to the coarse discretization in plotting the sets. Similar features will be seen in the next numerical example. The reliability feasible sets obtained by the brute-force analysis are shown in Figure 5, where the regions with label 'o' indicate the feasible region, while the label 'x' regions are infeasible. The comparison shows that the safety-factor feasible sets obtained by the new theorem are fairly close to the actual reliability feasible sets. The safety-factor feasible sets obtained by the old theorem do not perform terribly, but it is clear that the results from the new approach are superior to those from the old theorem.

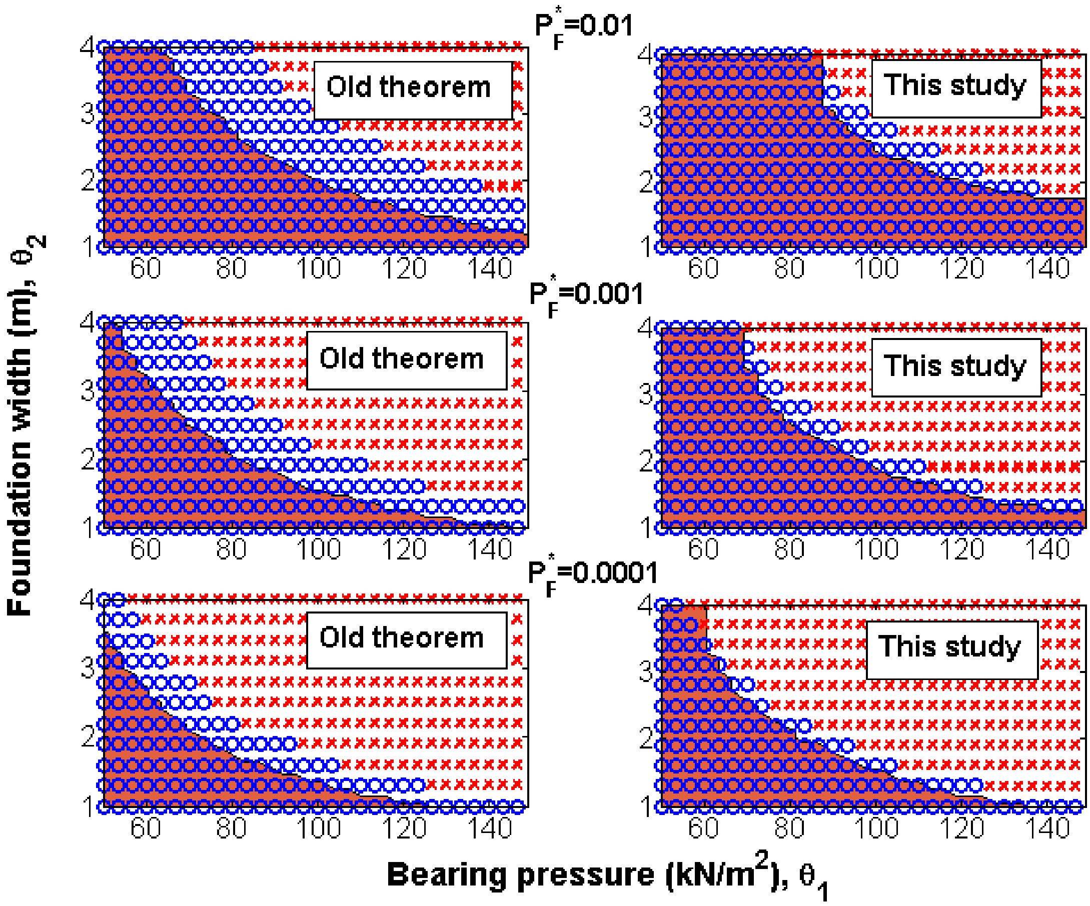

Figure 5 The safety-factor feasible sets for the consolidation example and their comparison with the actual reliability feasible sets (the region with label O); from top to bottom are for P_F^* = 0.01, 0.001, 0.0001.

5. Discussions and Remarks

1. The current method is originated from a method proposed by Ching (2009) that requires the distribution of G(Z,θ) to be invariant over θ. In some cases this requirement cannot be easily achieved. The contribution

of the current method is to find a series of $L^1_\theta, L^2_\theta, \ldots, L^m_\theta$ mappings to alleviate this restriction: now we only require the distribution of $L^k_\theta[G(Z,\theta)]$ to be invariant over θ. This does not mean that the $\check{R}(\theta)$ function can be chosen without care: the approximation of the new approach can be still quite bad if a poor $\check{R}(\theta)$ function is taken.

2. The performance of the proposed method mainly depends on two factors: (a) the feasibility of implementing a series of $L^1_\theta, L^2_\theta, \ldots, L^m_\theta$ mappings to make the distribution of $L^k_\theta[G(Z,\theta)]$ invariant; and (b) the accuracy of the stochastic simulation. The aspect (a) relies on the performance of GPD in fitting the tail of the G(Z,θ) distribution, while the aspect (b) can be easily improved by increasing the sample size *N*. It is empirically found that the choice of N = 2000~5000 will yield satisfactory results**.**
3. Since the proposed method is similar to subset simulation, it inherits most advantages of subset simulation: it is robust against uncertainty dimension and complexity of the target system.

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